MATH 231, Homework solutions #3

Part II

1. Consider the initial value problem

\[ x' = \frac{-2tx}{t^2 + \cos(x) + 2}, \quad x(0) = \pi \]

(a) Find the solution of the initial value problem in implicit form.

(b) Determine its interval of existence.

Solution.

(a) Rewrite the equation we obtain:

\[ 2tx + (t^2 + \cos(x) + 2)x' = 0 \]

By calculating \( M_x \) and \( N_t \), we get \( M_x = 2t = N_t \), so the equation is exact. We can find a function \( F(t, x) \) so that

\[
F_t = 2tx \\
F_x = t^2 + \cos(x) + 2
\]

Integrating the first equation, we obtain

\[ F(t, x) = t^2x + h(x) \]

Setting \( F_x = N \) gives

\[ F_x = t^2 + h'(x) = t^2 + \cos(x) + 2 \]

Thus \( h'(x) = \cos(x) + 2 \), and we can choose \( h(x) = \sin(x) + 2x \). So the general solution of the equation is given by \( F(t, x) = t^2x + \sin(x) + 2x = C \).

Substitute the initial condition \( x(0) = \pi \), we get the solution to the initial value problem is

\[ t^2x + \sin(x) + 2x = 2\pi \]

(b) To find its interval of existence, observe that the rate function

\[ f(t, x) = \frac{-2tx}{t^2 + \cos(x) + 2} \]

is continuous throughout the entire \( tx \)-plane. This is because the denominator \( t^2 + \cos(x) + 2 > 0 \) for any \((t, x)\). And \( \frac{\partial f}{\partial x} \) is also continuous throughout the entire \( tx \)-plane. Hence the solution to the initial value problem will be valid at least until it “leaves” the plane. Suppose the interval of existence for the solution is \((\alpha, \beta)\), if the endpoint (i.e. \( \alpha \) or \( \beta \)) is finite, then \( x(t) \) should go to infinity \((+\infty \text{ or } -\infty)\) as \( t \) approaches the endpoint.
However, from the solution $t^2 x + 2x + \sin(x) = 2\pi$, we know

$$x = \frac{2\pi - \sin(x)}{t^2 + 2}$$

which is always bounded:

$$|x| = \left| \frac{2\pi - \sin(x)}{t^2 + 2} \right| \leq \frac{2\pi + 1}{2}$$

i.e. the solution $x(t)$ will never go to infinity, thus the interval of existence must be $-\infty < t < \infty$.

2. For a second order differential equation with the dependent variable missing, i.e. an equation of the form $x'' = f(t, x')$, the substitution $v = x'$, $v' = x''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for $v$, then $x$ can be obtained by integrating $dx/dt = v$. Note that an arbitrary constant is obtained in solving the first order equation for $v$, and a second is introduced in the integration for $x$. Use this substitution to solve the second order differential equation:

$$t^2 x'' + 2tx' - 1 = 0, \quad t > 0$$

**Solution.**

Use the substitution given in the question, i.e. let $v = x'$, $v' = x''$, we get:

$$t^2 v' + 2tv - 1 = 0, \quad t > 0$$

This is a first order linear differential equation, and we can solve it for $v$. The general solution for $v$ is $v = 1/t + c_1/t^2$, where $c_1$ is an arbitrary constant. Integrating $x' = v$, we get $x = \ln(t) - c_1/t + c_2$. 