MATH 231, Homework Solutions #9

Part II

1. Suppose $A$ is a $2 \times 2$ real matrix and $\lambda_1 = \lambda_2 = r$ are its eigenvalues. Prove that if $A$ has two linearly independent eigenvectors, then $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$.

Proof.

Suppose $v_1$ and $v_2$ are two linearly independent eigenvectors of $A$, then

$$A v_1 = r v_1, \quad A v_2 = r v_2$$

Let $P$ be the $2 \times 2$ matrix whose columns are given by $v_1, v_2$, i.e.

$$P = (v_1 \ v_2)$$

thus

$$AP = A(v_1 \ v_2) = (Av_1 \ Av_2) = (rv_1 \ rv_2) = (v_1 \ v_2) \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = P \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

Since $v_1$ and $v_2$ are linearly independent, we know $\det P \neq 0$ thus $P$ is invertible. Therefore

$$A = P \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} P^{-1} = P(rI)P^{-1} = r(PIP^{-1}) = rI$$

where $I$ is the $2 \times 2$ identity matrix.

2. Find the general solution of the following system and sketch a phase portrait.

$$x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution.

Let us find the eigenvalues and eigenvectors for the coefficient matrix $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$.

The characteristic equation is given by:

$$\det(A - rI) = \det \begin{pmatrix} 2 - r & -5 \\ 1 & -2 - r \end{pmatrix} = (2 - r)(-2 - r) + 5 = r^2 + 1 = 0$$
The eigenvalues are \( r = \pm i \), which are complex numbers. Let \( r_1 = i \), its corresponding eigenvector \( \mathbf{v} \) is given by:

\[
(A - r_1I)\mathbf{v} = \mathbf{0}
\]

That is,

\[
\begin{pmatrix}
2 - i & -5 \\
1 & -2 - i
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

So an eigenvector is \( \mathbf{v} = \begin{pmatrix} 1 \\ 2 - i \end{pmatrix} \). Hence a complex-valued solution is

\[
x(t) = e^{it} \begin{pmatrix} 1 \\ 2 - i \end{pmatrix}
\]

\[
= (\cos t + i \sin t) \begin{pmatrix} 1 \\ 2 - i \end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos t + i \sin t \\
\left(\frac{2}{5} \cos t + \frac{1}{5} \sin t\right) + i\left(\frac{2}{5} \cos t - \frac{1}{5} \sin t\right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos t \\
\frac{2}{5} \cos t + \frac{1}{5} \sin t
\end{pmatrix} + i \begin{pmatrix}
\sin t \\
\frac{2}{5} \sin t - \frac{1}{5} \cos t
\end{pmatrix}
\]

We can choose the real and imaginary part of \( x(t) \), hence two real-valued solutions of the equation are

\[
\mathbf{u}(t) = \begin{pmatrix} \cos t \\ \frac{2}{5} \cos t + \frac{1}{5} \sin t \end{pmatrix}, \quad \mathbf{v}(t) = \begin{pmatrix} \sin t \\ \frac{2}{5} \sin t - \frac{1}{5} \cos t \end{pmatrix}
\]

and the general solution is:

\[
x(t) = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) = c_1 \begin{pmatrix} \cos t \\ \frac{2}{5} \cos t + \frac{1}{5} \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \frac{2}{5} \sin t - \frac{1}{5} \cos t \end{pmatrix}
\]

The eigenvalues of the coefficient matrix are complex with zero real part (i.e. pure imaginary), we know the origin is a center. Since the velocity vector at the point \((1, 0)\) is

\[
A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

which is pointing up, so the phase portrait is the following:
3. Find the general solution of the following system and sketch a phase portrait.

\[ \mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

**Solution.**

Let us find the eigenvalues and eigenvectors for the coefficient matrix \( A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \).

The characteristic equation is given by:

\[
\det(A - rI) = \det \begin{pmatrix} 1 - r & -5 \\ 1 & -3 - r \end{pmatrix} = (1 - r)(-3 - r) + 5 = r^2 + 2r + 2 = 0
\]

The eigenvalues are \( r = -1 \pm i \), which are complex numbers. Let \( r_1 = -1 + i \), its corresponding eigenvector \( \mathbf{v} \) is given by:

\[
(A - r_1 I) \mathbf{v} = \mathbf{0}
\]

That is,

\[
\begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
So an eigenvector is \( \mathbf{v} = \begin{pmatrix} 1 \\ -i/5 \end{pmatrix} \)

Hence a complex-valued solution is

\[
\mathbf{x}(t) = e^{(-1+i)t} \begin{pmatrix} 1 \\ -i/5 \end{pmatrix}
\]

\[
= e^{-t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ -i/5 \end{pmatrix}
\]

\[
= e^{-t} \begin{pmatrix} \cos t + i \sin t \\ (2/5 \cos t + 1/5 \sin t) + i(2/5 \sin t - 1/5 \cos t) \end{pmatrix}
\]

\[
= e^{-t} \begin{pmatrix} \cos t \\ 2/5 \cos t + 1/5 \sin t \end{pmatrix} + ie^{-t} \begin{pmatrix} \sin t \\ 2/5 \sin t - 1/5 \cos t \end{pmatrix}
\]

We can choose the real and imaginary part of \( \mathbf{x}(t) \), hence a set of real-valued solutions of the equation is

\[
\mathbf{u}(t) = e^{-t} \begin{pmatrix} \cos t \\ 2/5 \cos t + 1/5 \sin t \end{pmatrix}, \quad \mathbf{v}(t) = e^{-t} \begin{pmatrix} \sin t \\ 2/5 \sin t - 1/5 \cos t \end{pmatrix}
\]

and the general solution is:

\[
\mathbf{x}(t) = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2/5 \cos t + 1/5 \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ 2/5 \sin t - 1/5 \cos t \end{pmatrix}
\]

The eigenvalues of the coefficient matrix are complex with negative real part, we know the origin is a spiral point. In particular, it is a spiral sink. Since the velocity vector at the point \((1, 0)\) is \( A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) which is pointing up, the phase portrait is the following:
4. Find the general solution of the following system and sketch a phase portrait.

\[ x' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

**Solution.**

Let us find the eigenvalues and eigenvectors for the coefficient matrix \( A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \).

The characteristic equation is given by:

\[
\det(A - rI) = \det \begin{pmatrix} 2 - r & -1 \\ 1 & 4 - r \end{pmatrix} = r^2 - 6r + 9 = (r - 3)^2 = 0
\]

The eigenvalues are \( r_1 = r_2 = 3 \), which are repeated with multiplicity two. The eigenvector \( v_1 \) satisfies the equation:

\[
\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

We choose \( v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), so \( x_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) is a solution. To find another linearly independent solution, we consider the form:

\[ x_2(t) = te^{3t} v_1 + e^{3t} v_2 \]

where \( v_2 \) satisfies the equation:

\[
(A - 3I)v_2 = v_1
\]

That is,

\[
\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

We choose \( v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \).

So

\[ x_2(t) = te^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \]

is a solution. The general solution of the differential equation is

\[ x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 (te^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}) \]
The eigenvalues $r_1 = r_2 = 3 > 0$ and there is only one linearly independent eigenvector, so the origin is an improper node, in particular, it is an improper nodal source. Since the velocity vector at $(1, 0)$ is $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ which is pointing up, the phase portrait is the following:

![Phase portrait](image)

5. Consider the fourth order scalar differential equation:

$$x^{(4)} - x = 0$$

(a) Find the general solution of $x^{(4)} - x = 0$.

(b) Let $x_1 = x, x_2 = x', x_3 = x'', x_4 = x'''$, rewrite the scalar equation as a system of first order equations:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

(c) Use the result in part (a) to find the general solution of the linear system.

(d) Use the eigenvalue method to find the general solution of the linear system.

Solution.

(a) The characteristic equation is

$$r^4 - 1 = (r^2 + 1)(r^2 - 1) = 0$$
the roots are \( r = \pm i, \pm 1 \), thus four linearly independent solutions are

\[
X_1(t) = \cos(t), \quad X_2(t) = \sin(t), \quad X_3(t) = e^t, \quad X_4(t) = e^{-t}
\]

and the general solution is

\[
x(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 e^t + c_4 e^{-t}
\]

(b) We have \( x_1' = x_2, \ x_2' = x_3, \ x_3' = x_4 \) and \( x_4' = x^{(4)} = x = x_1 \), so the scalar equations is equivalent to

\[
\begin{pmatrix}
  x_1' \\
  x_2' \\
  x_3' \\
  x_4'
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix}
\]

(c) If \( x(t) \) is a solution of \( x^{(4)} - x = 0 \) then \( x(t) =
\begin{pmatrix}
  x(t) \\
  x'(t) \\
  x''(t) \\
  x'''(t)
\end{pmatrix}
\) is a solution for the linear system.

From (a) we immediately get four linearly independent solutions of the system:

\[
\begin{align*}
\mathbf{x}_1(t) &= \begin{pmatrix}
  \cos(t) \\
  -\sin(t) \\
  -\cos(t) \\
  \sin(t)
\end{pmatrix}, \\
\mathbf{x}_2(t) &= \begin{pmatrix}
  \sin(t) \\
  \cos(t) \\
  -\sin(t) \\
  -\cos(t)
\end{pmatrix}, \\
\mathbf{x}_3(t) &= \begin{pmatrix}
  e^t \\
  e^t \\
  e^t \\
  e^t
\end{pmatrix}, \\
\mathbf{x}_4(t) &= \begin{pmatrix}
  e^{-t} \\
  -e^{-t} \\
  e^{-t} \\
  -e^{-t}
\end{pmatrix}
\end{align*}
\]

Hence the general solution of the linear system is

\[
\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t) + c_4 \mathbf{x}_4(t)
\]
(d) Let \( A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \), then the characteristic equation is

\[
\det(A - rI) = \begin{vmatrix} -r & 1 & 0 & 0 \\ 0 & -r & 1 & 0 \\ 0 & 0 & -r & 1 \\ 1 & 0 & 0 & -r \end{vmatrix}
\]

\[
= (-r) \begin{vmatrix} -r & 1 & 0 \\ 0 & -r & 1 \\ 0 & 0 & -r \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & -r & 1 \\ 0 & 0 & -r \end{vmatrix} = r^4 - 1 = 0
\]

So the eigenvalues of \( A \) are \( r = \pm i, \pm 1 \).

For the complex eigenvalue \( r_1 = i \), its eigenvector satisfies

\[
(A - iI)v = 0
\]

That is,

\[
\begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

Solve this equation, we get \( b = ia, c = -a, d = -ia \), let \( a = 1 \), we get an eigenvector for \( r_1 = i \):

\[
v = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}
\]

So a complex-valued solution of the linear system is

\[
x(t) = e^{it}v = \begin{pmatrix} e^{it} \\ ie^{it} \\ -e^{it} \\ -ie^{it} \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ i(\cos t + i \sin t) \\ -(\cos t + i \sin t) \\ -i(\cos t + i \sin t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \\ -\cos(t) \\ \sin(t) \end{pmatrix} + i \begin{pmatrix} \sin(t) \\ -\cos(t) \\ \sin(t) \\ -\cos(t) \end{pmatrix}
\]
Thus $x_1(t) = \text{Re}(x)$ and $x_2(t) = \text{Im}(x)$ give us two linearly independent real-valued solutions.

For the eigenvalue $r = 1$, its eigenvector satisfies

$$(A - I)v = 0$$

That is,

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

Solve this equation, we get $b = a, c = a, d = -a$, let $a = 1$, we get an eigenvector for $r = 1$:

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus a solution of the linear system is

$$x_3(t) = e^t v = \begin{pmatrix} e^t \\ e^t \\ e^t \\ e^t \end{pmatrix}$$

For the eigenvalue $r = -1$, its eigenvector satisfies

$$(A - (-1)I)v = 0$$

That is,

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

Solve this equation, we get $b = -a, c = a, d = -a$, let $a = 1$, we get an eigenvector for $r = 1$:

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$
Thus a solution of the linear system is

\[ x_4(t) = e^{-t}v = \begin{pmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \\ -e^{-t} \end{pmatrix} \]

Hence the general solution of the linear system (using the eigenvalue method) is the same as what we have obtained in part (c).