Math 231, Tutorial solutions #4

1. Find the general solution of

\[ x'' + x = \tan t, \quad 0 < t < \frac{\pi}{2} \]

Hint:

\[ \frac{1}{\cos u} = \sec u, \quad \int \sec u \, du = \ln |\sec u + \tan u| \]

Solution.

We apply the method of variation of parameters,

\[ x(t) = u_1 \cos t + u_2 \sin t \]

So the system of equations for \( u_1', u_2' \) is

\[ u_1' \cos t + u_2' \sin t = 0 \]

\[ -u_1' \sin t + u_2' \cos t = \tan t \]

Which gives

\[ u_1' = -\frac{\sin^2 t}{\cos t} = -\frac{1 - \cos^2 t}{\cos t} = -\sec t + \cos t \]

\[ u_2' = \sin t \]

This gives \( u_1 = -\ln |\sec t + \tan t| + \sin t + C_1 \) and \( u_2 = -\cos t + C_2 \)

So the general solution is

\[ x(t) = u_1 \cos t + u_2 \sin t \]

\[ x(t) = -\cos t \ln |\sec t + \tan t| + C_1 \cos t + C_2 \sin t, \quad 0 < t < \frac{\pi}{2} \]
2. Verify that the given functions $x_1$ and $x_2$ satisfy the corresponding homogeneous equation; then find the general solution of the given non-homogeneous equation.

$$(1 - t)x'' + tx' - x = 2(t - 1)^2 e^{-t}, \quad 0 < t < 1, \quad x_1(t) = e^t, \quad x_2(t) = t$$

**Solution.**

We apply the method of variation of parameters. First we need the equation in standard form:

$$x'' + \frac{t}{1-t}x' - \frac{1}{1-t}x = 2(1 - t)e^{-t}, \quad 0 < t < 1$$

So the non-homogeneous term $g(t) = 2(1 - t)e^{-t}$.

The general solution is $x(t) = u_1 x_1 + u_2 x_2$, so the equations for $u_1', u_2'$ are

$$u_1' e^t + u_2' t = 0$$

$$u_1' e^t + u_2' = 2(1 - t)e^{-t}$$

which gives

$$u_1' = -2te^{-2t}, \quad u_2' = 2e^{-t}$$

This gives

$$u_1 = (t + \frac{1}{2})e^{-2t} + C_1, \quad u_2 = -2e^{-t} + C_2$$

So the general solution of the non-homogeneous equation is $x(t) = u_1 x_1 + u_2 x_2$

$$x(t) = (\frac{1}{2} - t)e^{-t} + C_1 e^t + C_2 t$$
3. A spring-mass system has a spring constant of 2 N/m. A mass of \( m \) kg is attached to the spring and a dashpot mechanism that has a damping constant of 1 kg·s/m. If the system is driven by an external force of 4 cos(2t) N,

(a) Determine the steady state response of this system.

(b) Find the value of the mass \( m \) for which the amplitude of the steady state response is maximum.

**Solution.**

(a) The equation for this system is

\[ mx'' + x' + 2x = 4 \cos(2t) \]

The steady state response is the particular solution we find by method of undetermined coefficients. That is \( x_p(t) = A \cos(2t) + B \sin(2t) \).

\[
x'_p(t) = -2A \sin(2t) + 2B \cos(2t)
\]
\[
x''_p(t) = -4A \cos(2t) - 4B \sin(2t)
\]

Plug in to the equation and compare coefficients we get

\[-4mA + 2B + 2A = 4\]
\[-4mB - 2A + 2B = 0\]

which gives

\[ A = \frac{1 - 2m}{2m^2 - 2m + 1}, \quad B = \frac{1}{2m^2 - 2m + 1} \]

So the steady state response is

\[ x_p(t) = \frac{1 - 2m}{2m^2 - 2m + 1} \cos(2t) + \frac{1}{2m^2 - 2m + 1} \sin(2t) \]

(b) The amplitude of the steady state response is

\[ R = \sqrt{A^2 + B^2} = \frac{\sqrt{2}}{\sqrt{2m^2 - 2m + 1}} \]

The amplitude \( R(m) \) reaches its maximum when the denominator \( \sqrt{2m^2 - 2m + 1} \) reaches its minimum; since \( 2m^2 - 2m + 1 = 2[(m - \frac{1}{2})^2 + \frac{1}{4}] \geq \frac{1}{2} \), we know \( R \) attains its maximum when \( m = \frac{1}{2} \).