1. Consider the initial value problem
\[ x' = 2tx^2, \quad x(a) = b \]

(a) What is the largest rectangle in the tx-plane for which the existence-uniqueness theorem is applicable?

(b) Solve the initial value problem, keeping the initial values a, b as parameters in the solution.

(c) When \( a = 0, b > 0 \), what is the largest interval that \( x(t, a, b) \) is valid? What happens to the solution as \( t \) approaches the endpoints of this interval?

**Solution.**

(a) The rate function of the differential equation is \( f(t, x) = 2tx^2 \). This function is continuous throughout the entire tx-plane. The derivative \( \frac{\partial f}{\partial x} = 4tx \) is continuous throughout the entire tx-plane. Thus the largest rectangle where the existence-uniqueness theorem apply would then be the entire tx-plane. The existence-uniqueness theorem then states, that for any initial condition \((a, b)\), there is a unique solution going through the point \((a, b)\) on some interval containing \(a\).

(b) First note that the constant function \( x(t) = 0 \) is a solution to the differential equation.

When \( x \neq 0 \), we can rewrite the equation as
\[ \frac{dx}{x^2} = 2tdt \]

which integrates to
\[ -\frac{1}{x} = t^2 + c \]

Substitute the initial condition \( x(a) = b \) to determine \( c \), we get
\[ x(t) = \frac{-1}{t^2 - a^2 - b^{-1}} \]

When \( b = 0 \), the solution to the initial value problem is the constant function \( x(t) = 0 \).

(c) When \( a = 0, b > 0 \), from the formula above, we get \( x(t) = \frac{-1}{t^2 - b} \) which is valid when \( t \neq \pm \sqrt{\frac{1}{b}} \). The initial time \( t = a (= 0) \) must belong to the interval of existence, so the maximal interval of existence is \(-\sqrt{\frac{1}{b}} < t < \sqrt{\frac{1}{b}} \). When \( t \) approaches the endpoints of its interval of existence, the value of \( x \to +\infty \).
2. Consider the initial value problem
\[ x' = \frac{1 + 3t^2}{3x(x - 2)}, \quad x(0) = 1 \]
(a) Find the solution of the initial value problem (implicit if necessary, explicit if convenient).
(b) Find the largest interval on which the solution is valid. \textit{Hint:} Find points where the solution curve has a vertical tangent line.

\textbf{Solution.}

(a) Separating variables:
\[ 3x(x - 2)dx = (1 + 3t^2)dt \]
The implicit solution is \[ x^3 - 3x^2 = t^3 + t + c. \] Use the initial condition \( x(0) = 1 \) to get \( c = -2 \). Thus the solution (in implicit form) to the initial value problem is \[ x^3 - 3x^2 = t^3 + t - 2 \]

(b) The implicit function gives a smooth curve in the \( tx \)-plane. The interval of validity of this solution extends on both sides of the initial point \((0, 1)\) as long as the function \( x(t) \) remains differentiable. The interval ends when we reach points where the tangent line is vertical. From the differential equation we know these are the points where \( 3x(x - 2) = 0 \), i.e. \( x = 0, 2 \).
When \( x = 0 \), substitute in the solution, we get \( t^3 + t - 2 = 0 \), so \( t = 1 \);
When \( x = 2 \), substitute in the solution, we get \( t^3 + t - 2 = -4 \), so \( t = -1 \).
Since the initial condition is at \( t = 0 \), we know the maximal interval of validity should be \(-1 < t < 1\).

Here is a picture of the implicit solution curve:
3. Solve the initial value problem

\[ x' = \frac{2 - e^t}{3 + 2x}, \quad x(0) = 0 \]

and determine where the solution attains its maximum value.

Solution.

This is a separable equation and it can be written as

\[ (3 + 2x)dx = (2 - e^t)dt \]

Integrating the left side with respect to \( x \) and the right side with respect to \( t \) gives

\[ 3x + x^2 = 2t - e^t + C \]

To determine \( C \) for the initial condition \( x(0) = 0 \), we substitute \( t = 0 \) and \( x = 0 \) and get \( C = 1 \). So the solution of the initial value problem in implicit form is given by

\[ 3x + x^2 = 2t - e^t + 1 \quad (1) \]

Observe that Eq. (1) is quadratic in \( x \), thus we can solve for \( x \) in terms of \( t \) explicitly:

\[ x = -\frac{3}{2} \pm \sqrt{2t - e^t + \frac{13}{4}} \]

Since the initial condition is \( x(0) = 0 \), we need to take the plus sign and the solution to the initial value problem is

\[ x(t) = -\frac{3}{2} + \sqrt{2t - e^t + \frac{13}{4}} \]

Based on the expression of \( x(t) \), we know it attains its maximum value when \( 2t - e^t + \frac{13}{4} \) does. Let \( h(t) = 2t - e^t + \frac{13}{4} \), then \( h(t) \) attains its maximum when \( t = \ln 2 \). Because \( t = \ln 2 \) is a critical point of \( h(t) \), i.e. \( h'(\ln 2) = 0 \). Moreover \( h''(\ln 2) = -2 < 0 \), so \( h(t) \) achieve its maximum at \( t = \ln 2 \).

Thus the solution \( x(t) \) of the initial value problem attains its maximum value at \( t = \ln 2 \).