Part I

Section 1.5, problem 36

A tank initially contains 60 gal of pure water. A mixture containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 hour.

(a) Find the amount of salt in the tank after $t$ minutes.

(b) What is the maximum amount of salt ever in the tank?

Solution.

(a) Let $x(t)$ be the amount of salt in the tank at time $t$, and we only consider time $t < 60$, then it satisfies the equation:

$$x' = 2 - \frac{3x}{60 - t}, \quad x(0) = 0, \quad t < 60$$

The equation in standard form is

$$x' + \frac{3x}{60 - t} = 2$$

the integrating factor is

$$\mu(t) = e^{\int \frac{3}{60 - t} dt} = e^{-3 \ln |60 - t|} = (60 - t)^{-3}$$

Multiply the integrating factor to the equation we get:

$$[(60 - t)^{-3} x]' = 2(60 - t)^{-3}$$

Since

$$\int 2(60 - t)^{-3} dt = (60 - t)^{-2}$$

The general solution is:

$$x(t) = \frac{(60 - t)^{-2} + C}{(60 - t)^{-3}} = (60 - t) + C(60 - t)^{3}$$

Substitute the initial condition $x(0) = 0$ we get $C = -1/60^2$. The amount of salt in the tank after $t$ minutes is

$$x(t) = (60 - t) - (60 - t)^{3}/3600$$
(b) To find the maximum value of \(x(t)\), we locate its critical points first:

\[ x'(t) = -1 + \frac{1}{1200}(60 - t)^2 = 0 \]

The critical point is \(t = 60 - \frac{10}{\sqrt{12}}\), and \(x''(60 - \frac{10}{\sqrt{12}}) < 0\), so \(x(t)\) achieves its maximum at \(t = 60 - \frac{10}{\sqrt{12}}\); the maximum amount of salt ever in the tank is

\[ x(60 - \frac{10}{\sqrt{12}}) = \frac{20}{3}\sqrt{12} \approx 23.09 \text{ lb} \]

Part II

1. Find the value of \(x_0\) for which the solution of the initial value problem

\[ x' - x = 1 + 3 \sin t, \quad x(0) = x_0 \]

remains finite as \(t \to \infty\).

**Solution.** This is a first-order linear equation in standard form. Computing the integrating factor, we get

\[ \mu(t) = e^{-t} \]

Multiply \(\mu(t)\) to both sides of the equation, we get

\[ e^{-t}x - e^{-t}x = e^{-t}(1 + 3 \sin t) \]

\[ \Rightarrow [e^{-t}x]' = e^{-t}(1 + 3 \sin t) \]

Thus

\[ e^{-t}x = \int e^{-t}(1 + 3 \sin t)dt + C = -e^{-t} + 3 \int e^{-t} \sin tdt + C \]

Computing

\[ \int e^{-t} \sin tdt = - \int \sin t d(e^{-t}) \]

\[ = -e^{-t} \sin t + \int e^{-t} \cos tdt \]

\[ = -e^{-t} \sin t - \int \cos t d(e^{-t}) \]

\[ = -e^{-t} \sin t - [e^{-t} \cos t + \int e^{-t} \sin tdt] \]

\[ = -e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \sin tdt \]

Hence \(\int e^{-t} \sin tdt = -(1/2)e^{-t}(\sin t + \cos t)\), and the general solution is:

\[ x(t) = -1 - (3/2)(\sin t + \cos t) + Ce^t \]
The initial condition is \( x(0) = x_0 \), so we get \( C = x_0 + 5/2 \), the solution to the initial value problem is

\[
x(t) = -1 - (3/2)(\sin t + \cos t) + (x_0 + 5/2)e^t
\]

Since \(-1 - (3/2)(\sin t + \cos t)\) will remain finite for any \( t \), and \( e^t \to \infty \) as \( t \to \infty \), we know the solution will remain finite when \( t \to \infty \) only if \( (x_0 + 5/2) = 0 \), i.e. \( x_0 = -5/2 \).

2. Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton’s law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 minute later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

**Solution.** Let \( T(t) \) be the temperature of the coffee at time \( t \), then it satisfies the equation:

\[
T' = -k(T - 70), \quad T(0) = 200
\]

where \( k \) is the conduction constant to be determined. The solution of the initial value problem is

\[
T(t) = 70 + 130e^{-kt}
\]

We know when \( t = 1 \) min, the temperature of the coffee is 190, hence \( T(1) = 190 \); use this condition to find the value of \( k \):

\[
T(1) = 70 + 130e^{-k} = 190 \quad \Rightarrow \quad k = \ln(13/12)
\]

So the temperature of the coffee at time \( t \) is given by

\[
T(t) = 70 + 130e^{-\left(\ln\frac{13}{12}\right)t}
\]

Let \( T(t) = 150 \), we get \( t = \ln\frac{13}{8}/\ln\frac{13}{12} \approx 6.07 \) min.
3. Consider the differential equation and initial condition

\[ \alpha t x + (3t^2 + 2\cos(x)\sin(x))x' = 0, \quad x(1) = \pi/4 \]

(a) Find a value of the parameter \( \alpha \) such that the equation is exact.

(b) For this value of \( \alpha \), find the solution of the initial value problem in implicit form.

Solution.

(a) By calculating \( M_x \) and \( N_t \), we get:

\[ M_x = \alpha t, \quad N_t = 6t \]

For the equation to be exact, we need \( M_x = N_t \), that is, \( \alpha = 6 \).

(b) Since the equation with \( \alpha = 6 \) is exact, we can find a function \( F(t, x) \) so that \( F_t = M, F_x = N \). i.e.

\[ F_t = 6tx \]
\[ F_x = 3t^2 + 2\cos(x)\sin(x) \]

Integrating the first equation, we obtain

\[ F(t, x) = 3t^2x + h(x) \]

Setting \( F_x = N \) gives

\[ F_x = 3t^2 + h'(x) = 3t^2 + 2\cos(x)\sin(x) \]

Thus \( h'(x) = 2\cos(x)\sin(x) \), and \( h(x) = \sin^2(x) + C_1 \). It is enough to choose any one of the \( h(x) \), and we choose \( h(x) = \sin^2(x) \). Therefore the general solution of the exact equation is given implicitly by

\[ F(t, x) = 3t^2x + \sin^2(x) = c \]

To determine the constant \( c \), we substitute the initial condition \( x(1) = \pi/4 \). So the solution to the initial value problem is

\[ 3t^2x + \sin^2(x) = 1/2 + 3\pi/4 \]