MTHE 235, Homework #6

Part II

1. Find the general solution using variation of parameters, where \( g(t) \) is a given continuous function.

\[ x'' + 4x = g(t). \]

Solution.

Two linearly independent solutions for the corresponding homogeneous equation are

\[ x_1 = \cos 2t, \quad x_2 = \sin 2t. \]

By variation of parameters, the general solution for the non-homogeneous equation is

\[ x = u_1 x_1 + u_2 x_2 \]

where \( u_1, u_2 \) satisfies the conditions:

\[
\begin{align*}
    u_1' \cos 2t + u_1' \sin 2t &= 0 \\
    -2u_1' \sin 2t + 2u_1' \cos 2t &= g(t)
\end{align*}
\]

Solve for \( u_1', u_2' \) we have

\[
\begin{align*}
    u_1' &= -\frac{1}{2}g(t) \sin 2t, \\
    u_2' &= \frac{1}{2}g(t) \cos 2t
\end{align*}
\]

So

\[
\begin{align*}
    u_1(t) &= \int_{t_0}^{t} -\frac{1}{2}g(s) \sin 2sd{s} + c_1; \\
    u_2(t) &= \int_{t_0}^{t} \frac{1}{2}g(s) \cos 2sd{s} + c_2
\end{align*}
\]

where \( t_0 \) is some initial time, \( c_1, c_2 \) are arbitrary constants. So the general solution for the non-homogeneous linear equation is:

\[
\begin{align*}
    x(t) &= u_1(t)x_1(t) + u_2(t)x_2(t) \\
    &= (\int_{t_0}^{t} -\frac{1}{2}g(s) \sin 2sd{s} + c_1) \cos 2t + (\int_{t_0}^{t} \frac{1}{2}g(s) \cos 2sd{s} + c_2) \sin 2t \\
    &= \frac{1}{2} \left[ \int_{t_0}^{t} (g(s)(-\sin 2s \cos 2t + \cos 2s \sin 2t))ds \right] + c_1 \cos 2t + c_2 \sin 2t \\
    &= \frac{1}{2} \int_{t_0}^{t} g(s) \sin[2(t - s)]ds + c_1 \cos 2t + c_2 \sin 2t
\end{align*}
\]

The last equality used the trigonometric identity: \( \sin A \cos B - \cos A \sin B = \sin(A - B) \).
2. Verify that the given functions $x_1, x_2$ satisfy the corresponding homogeneous equation; then find the general solution of the given non-homogeneous equation.

$$x'' - \frac{t + 2}{t}x' + \frac{t + 2}{t^2}x = 2t, \quad t > 0$$

$$x_1(t) = t$$

$$x_2(t) = te^t$$

**Solution.**

To verify $x_1, x_2$ are solutions for the corresponding homogeneous equation

$$x'' - \frac{t + 2}{t}x' + \frac{t + 2}{t^2}x = 0,$$

we substitute them to the equation. Straightforward calculation will show that $t$ and $te^t$ are solutions (Please check it).

We use the method of variation of parameters to solve the non-homogeneous equation. That is, we need to find two functions $u_1(t), u_2(t)$ which satisfies the conditions:

$$u_1'x_1 + u_2'x_2 = 0$$

$$u_1'x_1' + u_2'x_2' = 2t$$

and the solution of the non-homogeneous equation will be $x = u_1x_1 + u_2x_2$.

Use the provided solutions $x_1, x_2$ we get

$$u_1't + u_2'te^t = 0$$

$$u_1' + u_2'(t + 1)e^t = 2t$$

Solve for $u_1', u_2'$ we have

$$u_1' = -2, \quad u_2' = 2e^{-t},$$

Integrating $u_1', u_2'$, thus

$$u_1 = -2t + c_1, \quad u_2 = -2e^{-t} + c_2$$

Then the general solution of the non-homogeneous equation is given by

$$x = u_1x_1 + u_2x_2$$

$$= (-2t + c_1)t + (-2e^{-t} + c_2)te^t$$

$$= -2t^2 - 2t + c_1 t + c_2 te^t.$$