MTHE 235
Test #1
October 3, 2016
Instructor: Yanxia Deng

Name: ______________________

Student ID number: ______________

Instructions: This is a 50-minute exam. There are 3 questions worth a total of 30 points as indicated in the box below. Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise. No textbooks or notes. Only CASIO FX-991 or Gold/Blue Sticker calculators are permitted.

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Problem 1. \[ \frac{dx}{dt} = \frac{2t + 1}{e^x}, \quad x(0) = x_0 \]

(a) (7pts) Find the solution of the initial value problem explicitly.

(b) (3pts) For what values of \( x_0 \) is the solution \( x(t) \) defined for all \(-\infty < t < \infty\)?

(a) Separating variables,

\[ e^x \, dx = (2t+1) \, dt \]
\[ \Rightarrow \int e^x \, dx = \int (2t+1) \, dt + C \]
\[ \Rightarrow e^x = t^2 + t + C \]
\[ \Rightarrow x = \ln(t^2 + t + C) \]

\( x(0) = x_0 \), so \( x(0) = \ln(C) = x_0 \)

\[ \Rightarrow C = e^{x_0} \]

The solution is \( x(t) = \ln(t^2 + t + e^{x_0}) \).

(b) \( x(t) \) is defined if and only if \( t^2 + t + e^{x_0} > 0 \).

Since \( t^2 + t + e^{x_0} = (t+\frac{1}{2})^2 + e^{x_0} - \frac{1}{4} \)

it is positive for all \(-\infty < t < \infty\) only if \( e^{x_0} - \frac{1}{4} > 0 \).

So \( e^{x_0} > \frac{1}{4} \)

\[ \Rightarrow x_0 > \ln\left(\frac{1}{4}\right) \]
Problem 2. (10pts) Solve the initial value problem:

\[ tx' + 2x = \cos t, \quad x(\pi) = 0 \]

This is a 1st order linear equation, its standard form is

\[ x' + \frac{2}{t} x = \frac{\cos t}{t} \]

Integrating factor \( u = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = e^{\ln t^2} = t^2 \).

Multiply \( u \) to both sides:

\[ t^2 x' + 2 t x = t^2 \frac{\cos t}{t} \]

\[ \Rightarrow [t^2 x]' = t \cos t \]

So

\[ t^2 x = \int t \cos t \, dt + C \]

\[ = \int t \, d \sin t + C \]

\[ = t \sin t - \int \sin t \, dt + C \]

\[ = t \sin t + \cos t + C \]

Hence

\[ x = \frac{t \sin t + \cos t + C}{t^2} \]

Initial condition:

\[ x(\pi) = \frac{0 - 1 + C}{\pi^2} = 0 \]

\[ \Rightarrow C = 1 \]

So the solution is

\[ x(t) = \frac{t \sin t + \cos t + 1}{t^2} \]
Problem 3. Consider the differential equation and initial condition

\[(x^2 + 2 \cos t) + (\alpha xt + 4x^3)x' = 0, \quad x(0) = 1\]

(a) (3pts) Find the value of the parameter \(\alpha\) so that the equation is exact.

(b) (7pts) For this value of \(\alpha\), find the solution of the initial value problem in implicit form.

\(\text{a)} \quad M = x^2 + 2 \cos t, \quad N = \alpha xt + 4x^3\)

The equation is exact \(\iff M_x = N_t\)

since \(M_x = 2x, \quad N_t = \alpha x\)

so \(\alpha = 2\), the equation is exact.

\(\text{b)} \quad \text{When } \alpha = 2, \text{ the equation is exact, we can find } F(t, x)\)

so that \(\begin{cases} F_t = M = x^2 + 2 \cos t \quad (1) \\ F_x = N = 2xt + 4x^3 \quad (2) \end{cases}\)

From (1), \(F = x^2t + 2 \sin t + h(x)\)

so \(F_x = 2xt + h'(x) = 2xt + 4x^3\)

\(\implies h'(x) = 4x^3\)

\(h(x) = x^4 + C_1\)

we can choose \(h(x) = x^4\), so \(F(t, x) = x^2t + 2 \sin t + x^4\)

\(\implies\) the general solution is \(x^2t + 2 \sin t + x^4 = C\).

initial condition \(x(0) = 1\), so

\(0 + 0 + 1^4 = C \implies C = 1\)

The solution to the initial value problem is

\(x^2t + 2 \sin t + x^4 = 1\).\)