Similarity. Diagonalization

Math 112, week 11

Goals:

- Similarity of matrices.
- Diagonalizable matrices.

Suggested Textbook Readings: Sections §5.2, 5.3
Powers of a matrix.
Given a difference equation $\vec{x}_{k+1} = A\vec{x}_k$ with initial condition $\vec{x}_0$, then

$$\vec{x}_k = A^k \vec{x}_0$$

Example 1: If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, find $D^k$ for any $k \geq 1$.

Theorem: If $D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_n \end{bmatrix}$, then $D^k = \begin{bmatrix} d_1^k & 0 & 0 & 0 \\ 0 & d_2^k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_n^k \end{bmatrix}$
Example 2: If \( A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \), find \( A^k \) for any \( k \geq 1 \), given that \( A = PDP^{-1} \), where \( P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \), \( D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \).

**Definition (Similarity):** If \( A \) and \( B \) are \( n \times n \) matrices, then \( A \) is similar to \( B \) if there is an invertible matrix \( P \) such that \( P^{-1}AP = B \), or, equivalently, \( A = PBP^{-1} \).
**Theorem:** If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities.)

**Proof.**

**Note:**

1. Two matrices with the same eigenvalues is not sufficient to guarantee similarity.

For example, the matrices

\[
\begin{bmatrix}
2 & 1 \\
0 & 2 \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
2 & 0 \\
0 & 2 \\
\end{bmatrix}
\]

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If $A$ is row equivalent to $B$, then $B = EA$ for some invertible matrix.) Row operations on a matrix usually change its eigenvalues.
Definition: (Diagonalizable) A square matrix $A$ is said to be diagonalizable if $A$ is similar to a diagonal matrix. That is, $A = PDP^{-1}$

The Diagonalization Theorem: An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

Proof.
Note:

1. \( A = PDP^{-1} \), with \( D \) a diagonal matrix, if and only if the columns of \( P \) are \( n \) linearly independent eigenvectors of \( A \). In this case, the diagonal entries of \( D \) are eigenvalues of \( A \) that correspond, respectively, to the eigenvectors in \( P \).

2. In other words, \( A \) is diagonalizable if and only if there are enough eigenvectors to form a basis of \( \mathbb{R}^n \). We call such a basis an eigenvector basis of \( \mathbb{R}^n \).

Diagonalizing matrices.

Example 3: Diagonalize the following matrix, if possible.

\[
A = \begin{bmatrix}
1 & 3 & 3 \\
-3 & -5 & -3 \\
3 & 3 & 1
\end{bmatrix}
\]

That is, find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

Step 1. Find eigenvalues of \( A \).
Step 2. Find three linearly independent eigenvectors of $A$.

Step 3. Construct $P$ from the vectors in step 2.

Step 4. Construct $D$ from the corresponding eigenvalues.

Note: we can double check if $P$ and $D$ really work:

$AP =$

$PD =$
**Example 4: Diagonalize the following matrix, if possible.**

\[ A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \]
Independent Eigenvector Theorem: If $\vec{v}_1, \cdots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ of a square matrix $A$, then $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly independent.

Proof.

Theorem: An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.
Example 5: Determine if the following matrices are diagonalizable.

1. \[ A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & -2 \end{bmatrix} \]

2. \[ A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix} \]

3. \[ A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \]
**Theorem:** Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \cdots, \lambda_p$.

a. For $1 \leq k \leq n$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvalue $\lambda_k$.

b. The matrix $A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals $n$. 