Linear transformation and matrix operation
Math 112, week 4&5

Goals:

• Onto and one-to-one linear transformation.
• Matrix product, sum, transpose.
• Matrix Inverse.

Suggested Textbook Readings: Sections §1.9, §2.1, §2.2, §2.3
Example 1: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with standard matrix

$$
A = \begin{bmatrix}
1 & -4 & 8 & 1 \\
0 & 2 & -1 & 3 \\
0 & 0 & 0 & 5
\end{bmatrix}
$$

Does $T$ maps $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is $T$ one-to-one?

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $n > m$ can never be one-to-one.
Example 2: Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Does $T$ maps $\mathbb{R}^2$ onto $\mathbb{R}^3$? Is $T$ one-to-one?

Theorem: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $n < m$ can never map $\mathbb{R}^n$ onto $\mathbb{R}^m$. 
Example 3: Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

Does $T$ maps $\mathbb{R}^4$ onto $\mathbb{R}^4$? Is $T$ one-to-one?

Theorem: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one if and only if it maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. 
Example 4: Compute the matrix product $AB$, $BA$, if they are defined.

\[
A = \begin{bmatrix}
2 & 0 & -1 \\
1 & -3 & 2 \\
\end{bmatrix},
B = \begin{bmatrix}
1 & -2 \\
5 & 4 \\
0 & 1 \\
\end{bmatrix}
\]
Properties of matrix product.

Let $A$ be an $m \times n$ matrix, and $B, C$ are matrices with appropriate sizes.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_mA = A = AI_n$

Warning: in general,

1. $AB \neq BA$.
2. $AB = AC$ does not imply $B = C$.
3. $AB = 0$ does not imply $A = 0$ or $B = 0$.

Example 5: $(AB \neq BA)$

\[
A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}
\]
Powers of a matrix.

Let $A$ be a square matrix of order $n$ (i.e. size $n \times n$), then $A^k$ is the product of $k$ copies of $A$:

$$A^k = A \cdots A$$

$A^0$ is defined to be the identity matrix $I_n$.

Example 6: $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, compute $A^k$.

Transpose of a matrix.

Let $A$ be an $m \times n$ matrix, the transpose of $A$ is the $n \times m$ matrix, denoted by $A^T$, whose columns are the corresponding rows of $A$. 
Example 7: Find the transpose:

1. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2. $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

3. $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -1 & 6 \end{bmatrix}$

4. $D = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

Symmetric matrix: $A^T = A$
Properties of matrix transpose.

1. \((A^T)^T = A\)
2. \((A + B)^T = A^T + B^T\)
3. \((rA)^T = rA^T, \text{ } r \text{ is a scalar.}\)
4. \((AB)^T = B^T A^T\)

Example 8: Consider the number of roads connecting the four cities shown in the following diagram. Let \(a_{ij}\) be the number of possible roads connecting the \(i\)th city and the \(j\)th city without passing through another city. Represent this transit system by a matrix.
Now new roads are built as shown in the following diagram.

(a) Express the new roads in a matrix form, and find the total number of roads connecting any two cities without passing through another city.

(b) Find the number of ways to travel between any two cities by passing through exactly one city.

(c) Find the number of different round trips passing through exactly one city.
**Elementary matrix**: a matrix that is obtained by performing a single elementary row operation on $I_n$

**Example 9**: The following are elementary matrices.

1. $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Example 10**: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, compute $E_1A$, $E_2A$, $E_3A$. 
Example 11: Find the inverse of \( A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \).
The Invertible Matrix Theorem.
Let \( A \) be an \( n \times n \) matrix, then the following are equivalent.

1. \( A \) is an invertible matrix.
2. \( A \) is row equivalent to the \( n \times n \) identity matrix.
3. \( \text{rank}(A) = n \).
4. The equation \( A\vec{x} = \vec{0} \) has only the trivial solution.
5. The columns of \( A \) are linearly independent.
6. The linear transformation \( \vec{x} \mapsto A\vec{x} \) is one-to-one.
7. The linear system \( A\vec{x} = \vec{b} \) is consistent for each \( \vec{b} \) in \( \mathbb{R}^n \).
8. The columns of \( A \) span \( \mathbb{R}^n \).
9. The linear transformation \( \vec{x} \mapsto A\vec{x} \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).
10. There is an \( n \times n \) matrix \( C \) such that \( CA = I \).
11. There is an \( n \times n \) matrix \( D \) such that \( AD = I \).
12. \( A^T \) is an invertible matrix.
Example 12: Determine if the following matrices are invertible.

1. \[
\begin{bmatrix}
5 & 7 \\
-3 & -6 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
-4 & 6 \\
6 & -9 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
-7 & 0 & 4 \\
3 & 0 & -1 \\
2 & 0 & 9 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 3 & 7 & 4 \\
0 & 2 & 9 & 6 \\
0 & 0 & 1 & 8 \\
0 & 0 & 0 & 7 \\
\end{bmatrix}
\]