Dimension. Eigenvalue and eigenvector

Math 112, week 9

Goals:

- Bases, dimension, rank-nullity theorem.
- Eigenvalue and eigenvector.

Suggested Textbook Readings: Sections §4.5, 4.6, 5.1, 5.2
Find a basis for Col $A$.

**Theorem:** The columns with a leading entry in the RREF form a basis of Col $A$.

**Example:**

$$
\begin{bmatrix}
1 & 4 & 0 & 2 & -1 \\
3 & 12 & 1 & 5 & 5 \\
2 & 8 & 1 & 3 & 2 \\
5 & 20 & 2 & 8 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & 0 & 2 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

**Note:**

1. Choose the columns of $A$ itself for the basis of Col $A$. Row operation can change the column space of a matrix. The columns of an RREF $B$ of $A$ are often not in the column space of $A$.

2. If $A$ is a square $n \times n$ matrix and $\det A \neq 0$, then every linearly independent set with $n$ vectors in $\mathbb{R}^n$ form a basis for Col $A$. 
Theorem: If a vector space $V$ has a basis $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$, then any set in $V$ containing more than $n$ vectors cannot be linearly independent.

Proof.

Theorem: If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consists of exactly $n$ vectors.

Proof.
**Definition (Dimension):** The dimension of a vector space $V$, written as $\dim V$, is the number of vectors in a basis of $V$.

**Example 1:** Find the dimension of the vector space

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 2y + z = 0 \right\}$$

**Example 2:** Find the dimension of the vector space

$$V = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$
**Theorem:** If $H$ is a subspace of the vector space $V$, then
\[ \dim H \leq \dim V \]

**Proof.**

**Example 3:** The subspaces of $\mathbb{R}^3$ can be classified by dimension.

- **0-dimensional subspaces:**
  - 

- **1-dimensional subspaces:**
  - 

- **2-dimensional subspaces:**
  - 

- **3-dimensional subspaces:**
  - 
Example 4: For fixed integers $m, n$, the set $M_{m \times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

(1) Find the dimension of the space $M_{2 \times 2}$.

(2) Let $H$ be the set of all the diagonal matrices in $M_{2 \times 2}$, determine if $H$ is a subspace of $M_{2 \times 2}$. If $H$ is a subspace, find its dimension.
Dimension of Nul $A$ and Col $A$.

**Fact:** If $A$ is an $m \times n$ matrix, then

- The dimension of Nul $A$ is the number of free variables in the equation $A\vec{x} = \vec{0}$.

- The dimension of Col $A$ is equal to the number of leading entries in the RREF of $A$, i.e. rank $A$.

**Example 5:** Find the dimensions of the null space and the column space of the matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$
The Rank-Nullity Theorem:
Suppose $A$ is an $m \times n$ matrix, then

$$\text{Rank } A + \dim \text{Nul} A = n$$

Proof.

Example 6:

(1) If $A$ is a $7 \times 9$ matrix with a 2-dimensional null space, what is the rank of $A$?

(2) Could a $6 \times 9$ matrix have a two-dimensional null space?

(3) What is the maximal rank of a $11 \times 7$ matrix?
(4) Suppose two linearly independent solutions of $A\vec{x} = \vec{0}$, with 40 equations and 42 variables, are found, and all other solutions can be written as a linear combination of these two solutions. Based on this information, can you conclude that $A\vec{x} = \vec{b}$ is consistent for any right side value $\vec{b}$?

**The Invertible Matrix Theorem (continued):**

Let $A$ be an $n \times n$ matrix. Then each of the following statements is equivalent to the statement that $A$ is an invertible matrix.

1. The columns of $A$ form a basis of $\mathbb{R}^n$.
2. $\text{Col } A = \mathbb{R}^n$
3. $\dim \text{Col } A = n$
4. $\text{Rank } A = n$
5. $\text{Nul } A = \{ \vec{0} \}$
6. $\dim \text{Nul } A = 0$
Eigenvalues and Eigenvectors of a square matrix.

**Example 7:** Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the image of $\vec{u}, \vec{v}$ under multiplication by $A$.

**Definition (Eigenvalue and Eigenvector):** Let $A$ be an $n \times n$ matrix. The number $\lambda$ is an *eigenvalue* of $A$ if there exists a non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda \vec{v}$$

In this case, vector $\vec{v}$ is called an *eigenvector* of $A$ corresponding to $\lambda$. 
Example 8: Let \( A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \)

(a) Determine if \( \vec{u}, \vec{v} \) are eigenvectors of \( A \).

(b) Determine if 7 is an eigenvalue of \( A \).

Example 9: If \( \vec{v} \) is an eigenvector of \( A \) corresponding to \( \lambda \), what is \( A^4 \vec{v} \)? What about \( A^{100} \vec{v} \)?
**Theorem:** $\lambda$ is an eigenvalue of an $n \times n$ matrix if and only if the equation

$$(A - \lambda I) \vec{x} = \vec{0}$$

has a non-trivial solution.

**Proof.**

**Definition (Eigenspace):** Suppose $\lambda$ is an eigenvalue for $A$, then the eigenspace corresponding to $\lambda$ is the null space of the matrix $A - \lambda I$.

**Example 10:** *Find the eigenspaces corresponding to each eigenvalue of*

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$
Example 11: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of $A$ is 2; find a basis for the corresponding eigenspace.

Question: How can we find all the eigenvalues of a square matrix?

Example 12: Find all the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. 
The Characteristic Equation.

A scalar \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \) if and only if \( \lambda \) satisfies the characteristic equation

\[
\det(A - \lambda I) = 0
\]

Example 13: Find the characteristic equation of

\[
A = \begin{bmatrix}
5 & -2 & 6 & -1 \\
0 & 3 & -8 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 4 & 1
\end{bmatrix}
\]

Fact: If \( A \) is an \( n \times n \) matrix, then \( \det(A - \lambda I) \) is a polynomial of degree \( n \) called the characteristic polynomial of \( A \).

Fact: Every \( n \times n \) matrix has exactly \( n \) eigenvalues, counting multiplicity.

Example 14: The characteristic polynomial of a \( 6 \times 6 \) matrix is \( \lambda^6 - 4\lambda^5 - 12\lambda^4 \). Find the eigenvalues and their multiplicities.
The Invertible Matrix Theorem (continued):

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if 0 is not an eigenvalue of $A$.

Proof.

Process of finding the eigenvalues and eigenvectors.

1. Write down the characteristic equation $\det(A - \lambda) = 0$, and solve for $\lambda$.

2. For each $\lambda$, solve $(A - \lambda I)\vec{x} = \vec{0}$.

Example 15: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. 
Characteristic equation for $2 \times 2$ matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{tr}A = a + d, \quad \det A = ad - bc.$$ 

The characteristic polynomial of $A$ is equal to

$$\det(A - \lambda I) = \lambda^2 - (\text{tr}A)\lambda + \det A$$

Example 16: Suppose $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ has an eigenvalue $\lambda_1 = 3$, without solving the characteristic equation, find the other eigenvalue $\lambda_2$ of $A$. 