Math 112, Homework #6

Topics:

- Cramer’s rule, a formula for $A^{-1}$.
- Determinant with area and volume.
- Vector space, subspace, null space, column space.
- Linearly independent set.

Suggested Textbook Readings: Sections §3.3 §4.1-4.3

Question 1:

For the linear system:

$$
\begin{align*}
    x_1 + x_2 + 3x_3 &= 1 \\
    -2x_1 + 2x_2 + x_3 &= 0 \\
    x_2 + x_3 &= 1
\end{align*}
$$

1. Use Cramer’s rule to find the solution of the system.

2. Use the formula $A^{-1} = \frac{1}{\det A} \text{adj} A$ to find the inverse of the coefficient matrix, and then use this inverse to find the solution of the linear system.

Question 2: (The adjoint matrix)

1. Show that if $A$ is invertible, then $\text{adj} A$ is also invertible, and

   $$(\text{adj} A)^{-1} = \frac{1}{\det A} A$$

   (Hint: Given matrices $B$ and $C$, what relations between $B$, $C$ would show that $C$ is the inverse of $B$?)

2. Suppose $A$ is a $4 \times 4$ matrix with $\det A = -3$, find the determinant of $\text{adj} A$.

   (Hint: Find the relation between $A$ and $\text{adj} A$, and use the properties of determinant from Homework #5: if $B$ is an $n \times n$ matrix, then $\det B^{-1} = \frac{1}{\det B}$; $\det(rB) = r^n \det B$, where $r$ is a scalar.)

Question 3: (Determinant and area)

1. Find the area of the triangle with vertices $(0, 0), (5, 6), (-2, 7)$.

2. Find the area of the quadrilateral with vertices $(0, 0), (2, 3), (7, 1), (5, -2)$.
   (Hint: What relations between the four vertices will show that the quadrilateral is a parallelogram?)
Question 4: (Subspace.) For each of the following set \( W \), determine if \( W \) is a subspace of \( \mathbb{R}^3 \). Justify your answer.

1. \( W = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \} \).
2. \( W = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \} \).
3. \( W = \{ \begin{bmatrix} x + 3y - z \\ y + 2z \\ x + y \end{bmatrix} : x, y, z \in \mathbb{R} \} \).

Question 5: (Subspace.) Let \( V, W \) be subspaces of \( \mathbb{R}^3 \).

1. Is \( V \cap W \) a subspace? Here, \( V \cap W = \{ \vec{v} \in \mathbb{R}^3 : \vec{v} \in V \text{ and } \vec{v} \in W \} \), i.e. the set of vectors that are simultaneously in both \( V \) and \( W \).
2. Is \( V \cup W \) a subspace? Here, \( V \cup W = \{ \vec{v} \in \mathbb{R}^3 : \vec{v} \in V \text{ or } \vec{v} \in W \} \), i.e. the set of vectors that are either in \( V \) or \( W \).

Question 6: (Null space and column space) Express \( \text{Nul}A \) and \( \text{Col}A \) of each of the following matrices as the span of some vectors.

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1
\end{bmatrix},
A = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{bmatrix},
A = \begin{bmatrix}
1 & -2 & 0 & 4 & 0 \\
2 & -4 & 1 & -5 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]

Question 7: Let \( A = \begin{bmatrix}
-8 & -2 & 9 \\
6 & 4 & 8
\end{bmatrix} \) and \( \vec{u} = \begin{bmatrix}
2 \\
1 \\
-2
\end{bmatrix} \). Determine if \( \vec{u} \) is in \( \text{Col}A \). Is \( \vec{u} \) in \( \text{Nul}A \)?
Question 8: (Linearly independence and linearly dependence) 

Let \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

Let \( S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \} \).

(a) Explain why the set \( S \) is linearly dependent.

(b) Determine all ways to write the zero vector as a linear combination of \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \).

(Hint: Solve the system of linear equations: \( A\vec{x} = \vec{0} \), where the columns of \( A \) are \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \))

(c) Find a linearly independent subset of \( S \) that has the same span as \( S \).

(Hint: This is equivalent of finding a basis for the column space of \( A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \ \vec{v}_5] \))

(d) Can you find a matrix whose null space is the span of vectors \( \vec{x} \) from (b), and whose column space is the span of the vectors from (c)? How about a matrix whose column space is the span of the vectors from (c) but whose null space is the zero space?