1. Let \( a \) and \( b \) be positive integers and let \( b = qa + r \), where \( 0 \leq r < b \). Prove that
   
   (a) \( 2^b - 1 \equiv 2^r - 1 \pmod{2^a - 1} \);
   (b) \( \text{GCD}(2^a - 1, 2^b - 1) = 2^{\text{GCD}(a,b)} - 1 \).

2. In this problem we want to find the number of solutions of the equation \( x^2 = 1 \) in \( \mathbb{Z}_N \).
   (a) Find the number of solutions of \( x^2 = 1 \) in \( \mathbb{Z}_2, \mathbb{Z}_4 \), and in \( \mathbb{Z}_{2^k} \) for \( k \geq 3 \).
   
   (b) Find the number of solutions of \( x^2 = 1 \) in \( \mathbb{Z}_{p^k} \), where \( p \) is an odd prime and \( k \geq 1 \).
   
   (c) Find the number of solutions of \( x^2 = 1 \) in \( \mathbb{Z}_N \), where \( N = 2^k p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l} \). Here \( k \geq 0, k_1, k_2, \ldots, k_l \geq 1 \) and \( p_1, p_2, \ldots, p_l \) are distinct odd primes.
   
   **Hint.** Your answer should depend on \( k \) and \( l \) only. How did you use the Chinese Remainder Theorem?

3. Let \( \mathbb{F} \) be a field. Two variations of the ring of polynomial over \( \mathbb{F} \) are the **formal power series** \( \mathbb{F}[[X]] \) and the **formal Laurent series** \( \mathbb{F}((X)) \) defined as follows:
   
   \[
   \mathbb{F}[[X]] = \{ a_0 + a_1 X + a_2 X^2 + \ldots \mid a_i \in \mathbb{F} \}
   \]
   
   and
   
   \[
   \mathbb{F}((X)) = \{ a_{-k} X^{-k} + a_{-k+1} X^{-k+1} + a_{-k+2} X^{-k+2} + \ldots \mid k \in \mathbb{Z}, a_i \in \mathbb{F} \}.
   \]

   Show that both \( \mathbb{F}[[X]] \) and \( \mathbb{F}((X)) \) are integral domains. (Please, do not submit your work on this part!)
   
   (a) Prove that
   
   \[
   (\mathbb{F}[[X]])^\times = \{ a_0 + a_1 X + a_2 X^2 + \ldots \in \mathbb{F}[[X]] \mid a_0 \neq 0 \}.
   \]
   
   (b) Find the inverses of \( 1 + X \) and \( 1 - X - X^2 \) in \( \mathbb{F}[[X]] \).
   
   **Hint.** For \( 1 - X - X^2 \) you may need to recall the Fibonacci sequence from Problem 1 in Assignment 1.
   
   (c) Prove that \( \mathbb{F}((X)) \) is a field.