Problem Set #7  
Due: 2 November 2017

1. (a) Let \( R \) be the region \( 0 \leq y \leq x \) and \( 0 \leq x \leq 1 \). Evaluate \( \int_R (x+y) \, dA \) by making the change of variables \( x = u + v, \ y = u - v \). Check your answer by evaluating the integral directly.

(b) Use the change of variables \( x = u - uv, \ y = uv \), to calculate \( \int_R \frac{1}{x+y} \, dy \, dx \) where \( R \) is the region bounded by \( x = 0, \ y = 0, \ x + y = 1 \) and \( x + y = 4 \).

2. Suppose \( L \) is the line segment from the origin to the point \( (4,12) \) and \( \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \) is the vector field defined by \( \vec{F}(x,y) := xy\hat{i} + x\hat{j} \).

(a) Is line integral \( \int_L \vec{F} \cdot d\vec{r} \) greater than, less than, or equal to zero? Give a geometric explanation.

(b) A parameterization of \( L \) is \( \vec{\gamma} : [0,4] \to \mathbb{R}^2 \) where \( \vec{\gamma}(t) := t\hat{i} + 3t\hat{j} \). Use this to compute \( \int_L \vec{F} \cdot d\vec{r} \).

(c) Suppose a particle leaves the point \( (0,0) \), moves along the line towards the point \( (4,12) \), stops before reaching it and backs up, stops again and reverses direction, then completes its journey to the endpoint. All travel takes place along the line segment joining the point \( (0,0) \) to the point \( (4,12) \). If we call this path \( L' \), explain why \( \int_{L'} \vec{F} \cdot d\vec{r} = \int_L \vec{F} \cdot d\vec{r} \).

(d) A parameterization for a path like \( L' \) is given by \( \vec{\beta} : [0,4] \to \mathbb{R}^2 \) with \( \vec{\beta}(t) = \frac{1}{3}(t^3 - 6t^2 + 11t)\hat{i} + (t^3 - 6t^2 + 11t)\hat{j} \).

Check that this parameterization begins at the point \( (0,0) \) and ends at the point \( (4,12) \). Also check that all points of \( L' \) lie on the line segment connecting the point \( (0,0) \) to the point \( (4,12) \). What are the values of \( t \) at which the particle changes direction?

(e) Find \( \int_{L'} \vec{F} \cdot d\vec{r} \) using the parameterization in part (d).

3. Consider the vector field \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) where \( \vec{F}(x,y,z) := y\hat{i} + (3y^3-x)\hat{j} + z\hat{k} \). Evaluate \( \int_{C_n} \vec{F} \cdot d\vec{r} \) for the curve \( C_n \) parametrized by \( \vec{\gamma}_n : [0,1] \to \mathbb{R}^3 \) where \( \vec{\gamma}_n(t) := t\hat{i} + tn\hat{j} \).

What happens as \( n \to \infty \)?