Problem Set #11
Due: 4 December 2017
Note the extended deadline!

1. (a) Show that the path \( \vec{\gamma} : [0, 2\pi] \to \mathbb{R}^3 \) defined by \( \vec{\gamma}(t) := \cos(t)\vec{i} + \sin(t)\vec{j} + \sin(2t)\vec{k} \)
   lies on the surface \( z = 2xy \).
   (b) Evaluate \( \int_C (y^3 + \cos(x))dx + (\sin(y) + z^2)dy + xdz \) where \( C \) is the closed curve parametrized by \( \vec{\gamma} \).

2. (a) Evaluate the circulation of the vector field \( \vec{G}(x,y,z) := xy\vec{i} + z\vec{j} + 3y\vec{k} \) around a square of side length 6, centered at the origin lying in the \( yz \)-plane, and oriented counterclockwise viewed from the positive \( x \)-axis.
   (b) Let \( \vec{H}(x,y,z) := (y-z)\vec{i} + (x+z)\vec{j} + xy\vec{k} \) and let \( C \) be the circle of radius 3 centered at \((2,1,0)\) in the \( xy \)-plane oriented counterclockwise when viewed from above. Compute \( \int_C \vec{H} \cdot d\vec{r} \). Is \( \vec{H} \) path-independent? Explain.

3. Water in a bathtub has velocity vector field near the drain given, for \( x, y, z \) in cm, by
   \[
   \vec{V}(x,y,z) := \frac{-y\vec{i} + x\vec{j} + -z(x\vec{i} + y\vec{j})}{(z^2 + 1)^2} + \frac{\vec{k}}{z^2 + 1} = -\frac{y + xz}{(z^2 + 1)^2} \vec{i} - \frac{yz - x}{(z^2 + 1)^2} \vec{j} - \frac{1}{z^2 + 1} \vec{k} \text{ cm.s}^{-1}.
   \]
   (a) The drain in the bathtub is a disk in the \( xy \)-plane with center at the origin and radius 1 cm. Find the rate at which the water is leaving the bathtub.
   (b) Find the divergence of \( \vec{V} \).
   (c) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the \( xy \)-plane and oriented downward.
   (d) Consider the vector field
   \[
   \vec{U}(x,y,z) := \frac{1}{2} \left( \frac{y}{z^2 + 1} \vec{i} - \frac{x}{z^2 + 1} \vec{j} - \frac{x^2 + y^2}{(z^2 + 1)^2} \vec{k} \right).
   \]
   Compute \( \int_E \vec{U} \cdot d\vec{r} \) where \( E \) is the edge of the drain oriented clockwise when viewed from above.
   (e) Calculate \( \nabla \times \vec{U} \).
   (f) Explain why your answers in parts (a) and (d) are equal.

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1The denominators \((z^2 + 1)^2\) and \(z^2 + 1\) are always positive and so affect the magnitude (but not the direction) of the motion. The \((-y\vec{i} + x\vec{j})\) term represents rotation around the \(z\)-axis (counterclockwise when viewed from above). The \(-z(x\vec{i} + y\vec{j})\) term represents radial motion (towards the \(z\)-axis when \(z > 0\) and away when \(z < 0\)). The \(\vec{k}\) term is downward motion. Hence \(\vec{V}\) is a flow rotating inward and downward around the \(z\)-axis (for \(z > 0\)) like an actual bathtub drain.