Solutions #3

1. Find a vector field \( \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( \nabla \times \mathbf{F} = 2 \mathbf{i} - 3 \mathbf{j} + 4 \mathbf{k} \).

*Solution.* If \( \mathbf{r}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} + c \mathbf{k} \) for constants \( a, b, c \in \mathbb{R} \) then we have

\[
\mathbf{v} \times \mathbf{r} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a & b & c \\
x & y & z
\end{vmatrix}
= (bz - cy) \mathbf{i} - (az - cx) \mathbf{j} + (ay - bx) \mathbf{k}
\]

and

\[
\nabla \times (\mathbf{v} \times \mathbf{r}) = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
xz & -az + cx & ay - bx
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial}{\partial y} (ay - bx) - \frac{\partial}{\partial z} (cx - az)
\end{vmatrix} \mathbf{i}
- \begin{vmatrix}
\frac{\partial}{\partial x} (ay - bx) - \frac{\partial}{\partial z} (bz - cy)
\end{vmatrix} \mathbf{j}
+ \begin{vmatrix}
\frac{\partial}{\partial x} (cx - az) - \frac{\partial}{\partial y} (bz - cy)
\end{vmatrix} \mathbf{k}
= 2a \mathbf{i} + 2b \mathbf{j} + 2c \mathbf{k}
= 2 \mathbf{v}.
\]

Hence, if \( a = 1, b = -3/2 \) and \( c = 2 \) then

\[
\mathbf{F} = \mathbf{v} \times \mathbf{r} = (-3/2 z - 2y) \mathbf{i} + (2x - z) \mathbf{j} + (y + 3/2 x) \mathbf{k}
\]

satisfies \( \nabla \times \mathbf{F} = 2 \mathbf{i} - 3 \mathbf{j} + 4 \mathbf{k} \).

*Remark.* The vector potential \( \mathbf{F} \) is not uniquely determined. Since any gradient field is annihilated by the curl operator, \( \mathbf{F} + \nabla f \) is also a vector potential for any scalar field \( f : \mathbb{R}^3 \to \mathbb{R} \).

2. Find an equation for the plane tangent to the surface \( \sigma : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by

\[
\sigma(s, t) := e^s \mathbf{i} + t^2 e^{2s} \mathbf{j} + (2e^{-s} + t) \mathbf{k}
\]
at the point \((1, 4, 0)\).

*Solution.* Since \( \sigma(0, -2) = \mathbf{i} + 4 \mathbf{j} \) and

\[
\sigma_s(s, t) = \frac{\partial \sigma}{\partial s} = e^s \mathbf{i} + 2t^2 e^{2s} \mathbf{j} - 2e^{-s} \mathbf{k},
\sigma_s(0, -2) = \mathbf{i} + 8 \mathbf{j} - 2 \mathbf{k}
\]

\[
\sigma_t(s, t) = \frac{\partial \sigma}{\partial t} = 2te^{2s} \mathbf{j} + \mathbf{k},
\sigma_t(0, -2) = -4 \mathbf{j} + \mathbf{k},
\]

the standard normal vector arising from \( \sigma \) at \((0, -2)\) is

\[
\mathbf{N}(0, -2) = \sigma_s(0, -2) \times \sigma_t(0, -2) = \det \begin{bmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 8 & -2 \\
0 & -4 & 1
\end{bmatrix} = -\mathbf{j} - 4 \mathbf{k}.
\]
Hence, the tangent plane to the surface at $(1, 4, 0)$ is given by \( \vec{N} \cdot ((x-1)\vec{i}+(y-4)\vec{j}+z\vec{k}) = 0 \) or simply \( y + 4z = 4 \).

**3.** A torus (doughnut) is constructed by rotating a small circle of radius \( a \) in a large circle of radius \( b \) about the origin. The small circle is in a (rotating) vertical plane though the origin and the large circle is in the \( xy \)-plane.

Parameterize the torus as follows:

(a) Parameterize the large circle.

(b) For a typical point on the large circle, find two unit vectors which are perpendicular to one another and in the plane of the small circle at that point. Use these vectors to parameterize the small circle relative to its center.

(c) Combine your answers in the first two parts to parameterize the torus.

**Solution.**

(a) Since the large circle lies in the \( xy \)-plane, has radius \( b \), and is centered at the origin, it corresponds to the path \( \vec{\gamma}: [0, 2\pi) \rightarrow \mathbb{R}^3 \) given by \( \vec{\gamma}(\theta) := b \cos(\theta) \vec{i} + b \sin(\theta) \vec{j} \).

(b) A typical point on the large circle corresponds to the vector \( \vec{\gamma} = \vec{\gamma}(\theta) \) for a fixed value of \( \theta \). The unit vectors \( \frac{\vec{\gamma}}{||\vec{\gamma}||} \) and \( \vec{k} \) lie in the plane of the small circle. Since \( \vec{\gamma} \) is in the \( xy \)-plane, \( \frac{\vec{\gamma}}{||\vec{\gamma}||} \) is perpendicular to \( \vec{k} \). Thus, the small circle can be parameterized relative to its center by \( \vec{\beta}: [0, 2\pi) \rightarrow \mathbb{R}^3 \) where \( \vec{\beta}(\phi) := a \cos(\phi) \vec{\gamma} + a \sin(\phi) \vec{k} = a \cos(\phi) \cos(\theta) \vec{i} + a \cos(\phi) \sin(\theta) \vec{j} + a \sin(\phi) \vec{k} \).

(c) Since the center of the small circle is given by \( \vec{\gamma}(\theta) \), we obtain a parameterization of the torus by translating the parameterization in part (b). Specifically, we have the \( \vec{\tau}: [0, 2\pi) \times [0, 2\pi) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) given by \( \vec{\tau}(\theta, \phi) := \vec{\gamma}(\theta) + \vec{\beta}(\phi) = \cos(\theta)(b + a \cos(\phi)) \vec{i} + \sin(\theta)(b + a \cos(\phi)) \vec{j} + a \sin(\phi) \vec{k} \).

**Alternative Approach to 2(b).** The moving frame for \( \vec{\gamma}(\theta) \) is given by

\[
\begin{align*}
\vec{T}(\theta) &= \frac{\vec{\gamma}'(\theta)}{||\vec{\gamma}'(\theta)||} := \frac{-b \sin(\theta) \vec{i} + b \cos(\theta) \vec{j}}{b \sqrt{\cos^2(\theta) + \sin^2(\theta)}} = -\sin(\theta) \vec{i} + \cos(\theta) \vec{j} \\
\vec{N}(\theta) &= \frac{\vec{T}'(\theta)}{||\vec{T}'(\theta)||} := \frac{-\cos(\theta) \vec{i} - \sin(\theta) \vec{j}}{\sqrt{\cos^2(\theta) + \sin^2(\theta)}} = -\cos(\theta) \vec{i} - \sin(\theta) \vec{j} \\
\vec{B}(\theta) &= \vec{T} \times \vec{N} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & -\sin(\theta) & 0 \end{bmatrix} = \vec{k} ,
\end{align*}
\]
so the small circle lies in the plane spanned by $\vec{N}$ and $\vec{B}$. □