Solutions #6

1. Find the total area of the region inside the cardioid \( r = 1 - \cos(\theta) \) and outside the circle \( r = 1 \).

\[ \text{Solution.} \quad \text{Since} \quad \cos(\theta) \leq 0 \quad \text{and} \quad 1 - \cos(\theta) \geq 1 \quad \text{when} \quad \pi/2 \leq \theta \leq 3\pi/2, \quad \text{the region is} \quad R := \{(r, \theta) : 1 \leq r \leq 1 - \cos(\theta), \pi/2 \leq \theta \leq 3\pi/2 \}. \]

Hence, we have

\[
\text{Area} = \int_R 1 \, dA = \int_{\pi/2}^{3\pi/2} \int_1^{1-\cos(\theta)} r \, dr \, d\theta = \int_{\pi/4}^{3\pi/4} \left[ \frac{1}{2} r^2 \right]_1^{1-\cos(\theta)} d\theta
\]

\[
= \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left( (1 - \cos(\theta))^2 - 1 \right) \, d\theta
= \frac{1}{2} \int_0^{2\pi} -2 \cos(\theta) + \cos^2(\theta) \, d\theta
= \frac{1}{2} \left[ -2 \sin(\theta) + \frac{1}{2} (\cos(\theta) \sin(\theta) + \theta) \right]_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}. \quad \Box
\]

Remark. To compute \( \int \cos^2(\theta) \, d\theta \), we can use integration by parts:

\[
\int \cos^2(\theta) \, d\theta = \cos(\theta) \sin(\theta) + \int \sin^2(\theta) \, d\theta = \cos(\theta) \sin(\theta) + \int 1 - \cos^2(\theta) \, d\theta
= \cos(\theta) \sin(\theta) + \int 1 \, d\theta - \int \cos^2(\theta) \, d\theta = \cos(\theta) \sin(\theta) + \theta - \int \cos^2(\theta) \, d\theta.
\]

Hence, we have \( \int \cos^2(\theta) \, d\theta = \frac{1}{2} (\cos(\theta) \sin(\theta) + \theta) + C \). Alternatively, we can use the identity \( \cos^2(\theta) = \frac{1}{2} (1 + \cos(2\theta)) \):

\[
\int \cos^2(\theta) \, d\theta = \int \frac{1}{2} (1 + \cos(2\theta)) \, d\theta = \frac{1}{2} (\theta + \frac{1}{2} \sin(2\theta)) + C.
\]

Recall that \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \).

2. (a) Find the volume of an ice cream cone bounded by the cone \( z = \sqrt{x^2 + y^2} \) and the hemisphere \( z = \sqrt{8 - x^2 - y^2} \).

(b) Find the average distance to the origin for points in the ice cream cone region bounded by the hemisphere \( z = \sqrt{8 - x^2 - y^2} \) and the cone \( z = \sqrt{x^2 + y^2} \).

\[ \text{Solution.} \]
(a) The cone meets the hemisphere when \( \sqrt{x^2 + y^2} = \sqrt{8 - x^2 - y^2} \). In polar coordinates, this equation becomes \( r = \sqrt{8 - r^2} \iff 2r^2 = 8 \iff r = 2 \). Hence, we can compute the volume of the ice cream cone by finding the volume under the graph of \( \sqrt{8 - r^2} \) above the disk \( R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\} \) and subtracting the volume under the graph of \( r \) above \( R \). Therefore, we have

\[
\text{Volume} = \int_0^{2\pi} \int_0^2 (\sqrt{8 - r^2}) r \, dr \, d\theta - \int_0^{2\pi} \int_0^2 (r) r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 r \sqrt{8 - r^2} - r^2 \, dr \, d\theta = \int_0^{2\pi} \left[ -\frac{1}{3} (8 - r^2)^{3/2} - \frac{1}{3} r^3 \right]_0^2 \, d\theta
\]

\[
= \frac{1}{3} \int_0^{2\pi} (-4^{3/2} - 8 + 8^{3/2}) \, d\theta = \frac{1}{3} (16\sqrt{2} - 16) \int_0^{2\pi} d\theta
\]

\[
= \frac{1}{3} (32\pi)(\sqrt{2} - 1).
\]

(b) In spherical coordinates, the hemisphere is given by

\[
\rho \cos(\phi) = \sqrt{8 - \rho^2 \sin^2(\phi) \cos^2(\theta) - \rho^2 \sin^2(\phi) \sin^2(\theta)}
\]

\[
\rho^2 \cos^2(\phi) = 8 - \rho^2 \sin^2(\phi)
\]

\[
\rho = 2\sqrt{2}
\]
and the cone is given by the equation
\[ \rho \cos(\phi) = \sqrt{\rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta)} \]
\[ \rho \cos(\phi) = \rho \sin(\phi) \]
\[ \tan(\phi) = 1 \implies \phi = \frac{\pi}{4}. \]
Hence, the ice cream cone is the region
\[ W := \{(\rho, \theta, \phi) : 0 \leq \rho \leq 2\sqrt{2}, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4 \}. \]
Since the function which measures the distance from the origin to a point is simply \( \rho \), the average distance to the origin for points in \( W \) is:
\[
\text{Average} = \frac{1}{\text{Volume}(W)} \int_W \rho \, dV = \frac{1}{\text{Volume}(W)} \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \rho \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta
\]
\[ = \frac{1}{\text{Volume}(W)} \int_0^{2\pi} \int_0^{\pi/4} \left[ \frac{1}{4} \rho^4 \sin(\phi) \right]_0^{2\sqrt{2}} \, d\phi \, d\theta
\]
\[ = \frac{32\pi}{\text{Volume}(W)} \left[ -\cos(\phi) \right]_0^{\pi/4} = \frac{32\pi(1 - \frac{1}{\sqrt{2}})}{\text{Volume}(W)}\]
Moreover, we have [also see part (a)]:
\[
\text{Volume}(W) = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{32\pi(\sqrt{2} - 1)}{3}. \]
Therefore, the average distance to the origin for points in the ice cream cone region \( R \) is \( 3/\sqrt{2} \). \( \Box \)

3. (a) A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm. Set up a triple integral in cylindrical coordinates representing the volume of the bead. Evaluate the integral.
(b) A half-melon is approximated by the region between two concentric hemispheres, one of a radius \( a \) and the other of radius \( b \) with \( 0 < a < b \). Write a triple integral, including limits of integration, giving the volume of the half-melon. Evaluate the integral.

Solution.

(a) In cylindrical coordinates, the sphere is given by the equation \( r^2 + z^2 = 25 \) and the hole is given by \( r = 1 \). Hence, the bead is the region
\[ B := \{(r, \theta, z) : -\sqrt{25 - r^2} \leq z \leq \sqrt{25 - r^2}, 0 \leq \theta \leq 2\pi, 1 \leq r \leq 5 \}. \]
We find the volume by integrating the constant density function 1 over $B$:

\[
\text{Volume} = \int_B 1 \, dV = \int_1^5 \int_0^{2\pi} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r \, dz \, d\theta \, dr = \int_1^5 \int_0^{2\pi} \left[ \frac{r^2}{\sqrt{25-r^2}} \right]_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} d\theta \, dr \\
= \int_1^5 2r\sqrt{25-r^2} \int_0^{2\pi} d\theta \, dr = 2\pi \left[ -\frac{2}{3} (25-r^2)^{3/2} \right]_1^5 = 64\pi \sqrt{6} \text{ mm}^3.
\]

(b) In spherical coordinates, the outer hemisphere is the region

\[
O := \{ (\rho, \phi, \theta) : 0 \leq \rho \leq b, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi \},
\]
and the inner hemisphere is the region

\[
I := \{ (\rho, \phi, \theta) : 0 \leq \rho \leq a, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi \},
\]
so the half-melon is the region

\[
M := O \setminus I = \{ (\rho, \phi, \theta) : a \leq \rho \leq b, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi \}.
\]

Hence, the volume of the half-melon is

\[
\text{Volume} = \int_M 1 \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \\
= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\pi/2} \sin(\phi) \, d\phi \right) \left( \int_a^b \rho^2 \, d\rho \right) \\
= (2\pi) \left[ -\cos(\phi) \right]_0^{\pi/2} \left[ \frac{1}{3} \rho^3 \right]_a^b = \frac{2\pi}{3} (b^3 - a^3).
\]

\[\square\]