1. (a) Let $R$ be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate $\int_R (x + y) \, dA$ by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly.

(b) Use the change of variables $x = u - uv$, $y = uv$, to calculate $\int_R \frac{1}{x+y} \, dy \, dx$ where $R$ is the region bounded by $x = 0, y = 0, x + y = 1$ and $x + y = 4$.

Solution.

(a) The change of variables $x = u + v$, $y = u - v$ maps the lines $y = 0$, $x = 1$ and $y = x$ into $v = u$, $v = -u + 1$ and $v = 0$ respectively. Hence, the region $B$ in the $xy$-plane corresponds to the region $T = \{(u, v) : v \leq u \leq -v + 1, 0 \leq v \leq 1/2\}$. Since

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2$$

we have

$$\int_B (x + y) \, dx \, dy = \int_T ((u + v) + (v - u)) \, du \, dv = \int_0^{1/2} \int_v^{-v+1} 4u \, du \, dv$$

$$= 2 \int_0^{1/2} \left[ u^2 \right]_v^{-v+1} \, dv = 2 \int_0^{1/2} -2v + 1 \, dv = 2 \left[ -v^2 + v \right]_0^{1/2} = \frac{1}{2}.$$

Evaluating the integral directly, we also obtain

$$\int_B (x + y) \, dA = \int_0^1 \int_0^x (x + y) \, dy \, dx = \int_0^1 \left[ xy + \frac{1}{2} y^2 \right]_0^x \, dx$$

$$= \int_0^1 \frac{3}{2} x^2 \, dx = \left[ \frac{1}{2} x^3 \right]_0^1 = \frac{1}{2}.$$

(b) The change of variables $x = u - uv$, $y = uv$ maps the lines $x = 0$, $y = 0$, $x + y = 1$ and $x + y = 4$ into $u(1 - v) = 0$, $uv = 0$, $u = 1$ and $u = 4$ respectively. Hence, the region $R$ corresponds to the region $S = \{(u, v) : 1 \leq u \leq 4, 0 \leq v \leq 1\}$. Since

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 - v & -u \\ v & u \end{bmatrix} = (1 - v)u + uv = u,$$

we have

$$\int_R \frac{1}{x+y} \, dA = \int_S 1/u \, du \, dv = \int_1^4 \int_0^1 dv \, du = (4 - 1)(1) = 3.$$

2. Suppose $L$ is the line segment from the origin to the point $(4, 12)$ and $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field defined by $\vec{F}(x, y) := xy \vec{\imath} + x \vec{\jmath}$.

(a) Is line integral $\int_L \vec{F} \cdot d\vec{r}$ greater than, less than, or equal to zero? Give a geometric explanation.

(b) A parameterization of $L$ is $\vec{r} : [0, 4] \to \mathbb{R}^2$ where $\vec{r}(t) := t \vec{\imath} + 3t \vec{\jmath}$. Use this to compute $\int_L \vec{F} \cdot d\vec{r}$.
(c) Suppose a particle leaves the point \((0, 0)\), moves along the line towards the point \((4, 12)\), stops before reaching it and backs up, stops again and reverses direction, then completes its journey to the endpoint. All travel takes place along the line segment joining the point \((0, 0)\) to the point \((4, 12)\). If we call this path \(L'\), explain why \(\int_{L'} \vec{F} \cdot d\vec{r} = \int_L \vec{F} \cdot d\vec{r}\).

(d) A parameterization for a path like \(L'\) is given by \(\vec{\beta}: [0, 4] \to \mathbb{R}^2\) with

\[
\vec{\beta}(t) = \frac{1}{3}(t^3 - 6t^2 + 11t)\hat{i} + (t^3 - 6t^2 + 11t)\hat{j}.
\]

Check that this parameterization begins at the point \((0, 0)\) and ends at the point \((4, 12)\). Also check that all points of \(L'\) lie on the line segment connecting the point \((0, 0)\) to the point \((4, 12)\). What are the values of \(t\) at which the particle changes direction?

(e) Find \(\int_{L'} \vec{F} \cdot d\vec{r}\) using the parameterization in part (d).

Solution.

(a) On the rectangle \(0 \leq x \leq 4\) and \(0 \leq y \leq 12\), both components of the vector field \(\vec{F}\) are positive. Since the vector field is roughly pointing in the same direction as the path, we expect the line integral \(\int_L \vec{F} \cdot d\vec{r}\) to be positive.

(b) We have

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^4 \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) \, dt = \int_0^4 (3t^3\hat{i} + t\hat{j}) \cdot (\hat{i} + 3\hat{j}) \, dt
\]

\[
= \int_0^4 3t^2 + 3t \, dt = \left[t^3 + \frac{3}{2}t^2\right]_0^4 = 88.
\]

(c) Let \(P\) denote the first point on the line at which the particle stops and let \(Q\) denote the second. The path \(L'\) can be divided into four parts:

- \(C_1\) is the path from the origin to \(P\);
- \(C_2\) is the path from \(P\) to \(Q\);
- \(C_3\) is the path from \(Q\) to \(P\);
- \(C_4\) is the path from \(P\) to \((4, 12)\).

Since all the travel takes place along the line segment joining the point \((0, 0)\) to the point \((4, 12)\), we have \(C_2 = -C_3\) and \(C_1 + C_4 = L\). Thus, we have

\[
\int_{L'} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}
\]

\[
= \int_{C_1+C_4} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}
\]

\[
= \int_{C_1+C_4} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \int_L \vec{F} \cdot d\vec{r}.
\]
(d) Since \( \vec{\beta}(0) = \vec{0} \) and
\[
\vec{\beta}(4) = \frac{1}{3}(3^3 - 6 \cdot 4^2 + 11 \cdot 4)\vec{r} + (4^3 - 6 \cdot 4^2 + 11 \cdot 4)\vec{J} \\
= \frac{4}{3}(16 - 24 + 11)\vec{r} + 4(16 - 24 + 11)\vec{J} = 4\vec{r} + 12\vec{J},
\]
this parameterization begins at the point \((0,0)\) and ends at the point \((4,12)\). The equation of the line through \((0,0)\) and \((4,12)\) is \(3x - y = 0\). Because \( \vec{\beta}(t) \cdot (3\vec{r} - \vec{J}) = 0 \), it follows that all points of \( L' \) lie on the line segment connecting the point \((0,0)\) to the point \((4,12)\). Finally, the particle changes direction when its speed is zero. Since \( \vec{\beta}'(t) = \frac{1}{3}(3t^2 - 12t + 11)\vec{r} + (3t^2 - 12t + 11)\vec{J} \), this occurs with \( 3t^2 - 12t + 11 = 0 \) and the quadratic formula implies that
\[
t = \frac{12 \pm \sqrt{144 - 12(3)(11)}}{2(3)} = \frac{12 \pm \sqrt{2}}{6} = 2 \pm \frac{1}{\sqrt{3}}.
\]
Therefore, the particle changes direction at time \( 2 - \frac{1}{\sqrt{3}} \) and \( 2 + \frac{1}{\sqrt{3}} \).

(e) We have
\[
\int_{L'} \vec{F} \cdot d\vec{r} = \int_0^4 \vec{F}(\vec{\beta}(t)) \cdot \vec{\beta}'(t) \, dt \\
= \int_0^4 \left( \frac{t^3 - 6t^2 + 11t}{3} \vec{r} + \frac{t^3 - 6t^2 + 11t}{3} \vec{J} \right) \cdot \left( \frac{3t^2 - 12t + 11}{3} \vec{r} + (3t^2 - 12t + 11) \vec{J} \right) \, dt \\
= \int_0^4 \left( \frac{t^3 - 6t^2 + 11t}{9} + \frac{t^3 - 6t^2 + 11t}{9} \right) (3t^2 - 12t + 11) \, dt.
\]
If \( w = t^3 - 6t^2 + 11t \) then \( dw = (3t^2 - 12t + 11) \, dt \) and
\[
\int_{L'} \vec{F} \cdot d\vec{r} = \int_0^{12} \frac{w^2}{9} + \frac{w^3}{3} \, dw = \left[ \frac{w^3}{27} + \frac{w^4}{6} \right]_0^{12} = 88. \quad \square
\]

3. Consider \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) where \( \vec{F}(x,y,z) := y\vec{i} + (3y^3 - x)\vec{j} + z\vec{k} \). Evaluate \( \int_{C_n} \vec{F} \cdot d\vec{r} \) for the curve \( C_n \) parametrized by \( \vec{\gamma} : [0,1] \to \mathbb{R}^3 \) where \( \vec{\gamma}(t) := t\vec{i} + t^n\vec{J} \). What happens as \( n \to \infty \)?

**Solution.** We have
\[
\int_{C_n} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t,t^n,0) \cdot (\vec{r} + nt^{n-1}\vec{J}) \, dt = \int_0^1 (t^n\vec{r} + (3t^3 - t)\vec{J}) \cdot (\vec{r} + nt^{n-1}\vec{J}) \, dt \\
= \int_0^1 3nt^{4n-1} + (1 - n)t^{n} \, dt = \left[ \frac{3nt^{4n}}{4n} + \frac{1-n}{n+1}t^{n+1} \right]_0^1 = \frac{3}{4} + \frac{1-n}{n+1}.
\]
Hence, we have \( \lim_{n \to \infty} \int_{C_n} \vec{F} \cdot d\vec{r} = \lim_{n \to \infty} \left( \frac{3}{4} + \frac{1-n}{n+1} \right) = \frac{3}{4} - 1 = -\frac{1}{4}. \quad \square \)