Solutions #9
MATH 387 : 2017

1. In Cartesian plane over the ordered field \( \mathbb{F} \), consider an angle \( \alpha \) formed by two rays lying on lines of slope \( m \) and \( m' \). The tangent of \( \alpha \) is defined to be

\[
\tan(\alpha) = \pm \left| \frac{m' - m}{1 + m \cdot m'} \right|,
\]

where we take the positive sign if the angle is acute and the negative sign if the angle is obtuse. Using this definition, verify that for any two acute angles \( \alpha \) and \( \beta \), we have

\[
\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \cdot \tan(\beta)}.
\]

**Solution.** Consider three rays emanating from a point \( A \) such that the angle formed by the two adjacent rays is \( \alpha \) and the angle formed by the another two adjacent rays is \( \beta \). Let \( m, m', \) and \( m'' \) denote the slopes of the three rays where \( m' \) is the slope of the ray common to both \( \alpha \) and \( \beta \).

![Diagram of three rays forming angles](image)

If we have \( m < m' < m'' \), then we have \( 1 + m \cdot m' > 0 \) and \( 1 + m' \cdot m'' > 0 \) because the angles \( \alpha \) and \( \beta \) are acute. Similarly, when \( m'' < m < m' \), we have \( 1 + m \cdot m' > 0 \) and \( 1 + m' \cdot m'' < 0 \), and when \( m' < m'' < m \), we have \( 1 + m \cdot m' < 0 \) and \( 1 + m' \cdot m'' > 0 \). Hence, in all three cases, we obtain

\[
\frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \cdot \tan(\beta)} = \frac{m' - m}{1 + m \cdot m'} + \frac{m'' - m'}{1 + m' \cdot m''},
\]

\[
= \frac{(m' - m) \cdot (1 + m' \cdot m'' + (m'' - m') \cdot (1 + m \cdot m')} \]

\[
= \frac{1 + m \cdot m'}{1 + m' \cdot m'' \cdot (m' - m) \cdot (m'' - m')}
\]

\[
= \frac{m \cdot m^2 + m^2 \cdot m' \cdot m'' - m + m''}{(1 + m \cdot m') \cdot (m'' - m')}
\]

\[
= \frac{m'' - m}{1 + m \cdot m''}
\]

\( \square \)

**Remark.** If \( \alpha + \beta \) is a right angle, then \( m \cdot m'' = -1 \) and \( \tan(\alpha + \beta) = \infty \). If \( \alpha + \beta \) is obtuse, then either \( m'' < m \) or \( 1 + m \cdot m'' < 0 \), so \( \tan(\alpha + \beta) < 0 \).

2. In the Cartesian plane over an ordered field \( \mathbb{F} \), consider a right triangle \( ABC \) where \( \angle ABC \) is a right angle. Let \( D \) and \( E \) be the midpoints of the segments \( AB \) and \( AC \) respectively. Show that there exists a line segment \( FG \) such that \( F \) is between \( B \) and \( D \), \( G \) is between \( C \) and \( E \), \( FG \) is parallel to \( BC \), and \( EF \) is parallel to \( BG \) if and only if \( \sqrt{2} \in \mathbb{F} \).
Solution. Assume first that a segment $FG$ as required exists. Let $k := \frac{AB}{AF} \in \mathbb{F}$. The pairs of triangles $(ABG, AFE)$ and $(ABC, AFG)$ are similar which implies

$$k = \frac{AB}{AF} = \frac{AG}{AE} \quad \text{and} \quad k = \frac{AB}{AF} = \frac{AC}{AG}.$$  

Multiplying the two equalities above we obtain

$$k^2 = \frac{AG}{AE} \cdot \frac{AC}{AG} = \frac{AC}{AE} = 2$$

which implies that $k = \sqrt{2}$, i.e., $\sqrt{2} \in \mathbb{F}$.

Conversely, if $\sqrt{2} \in \mathbb{F}$, choose $F$ between $A$ and $B$ so that $\frac{AB}{AF} = \sqrt{2}$ and choose $G$ between $A$ and $C$ so that $FG$ is parallel to $BC$. Then

$$\frac{AC}{AG} = \frac{AB}{AF} = \sqrt{2} \quad \text{and} \quad \frac{AC}{AE} = 2.$$  

Dividing the two equations above we obtain

$$\sqrt{2} = \frac{AC}{AE} : \frac{AC}{AG} = \frac{AG}{AE}.$$  

In particular, $\frac{AG}{AE} = \frac{AB}{AF}$ and hence the triangles $ABG$ and $AFE$ are similar by (Sim SAS). Thus $\angle ABG = \angle AF E$ which proves that $BG$ is parallel to $FE$. \hfill \Box

3. For a spherical triangle $ABC$, prove the following half-angle formulas

$$\sin \left( \frac{\sigma}{2} \right) = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin(b) \sin(c)}} \quad \cos \left( \frac{\sigma}{2} \right) = \sqrt{\frac{\cos(\beta-\gamma) \cos(\gamma-\beta)}{\sin(\beta) \sin(\gamma)}}$$

where $s := \frac{1}{2}(a+b+c)$ and $\sigma := \frac{1}{2}(\alpha+\beta+\gamma)$.

Solution. First, the addition formula for cosine yields

$$\cos(\phi) - \cos(\theta) = \cos \left( \frac{\theta + \phi}{2} - \frac{\theta - \phi}{2} \right) - \cos \left( \frac{\theta + \phi}{2} + \frac{\theta - \phi}{2} \right)$$

$$= \cos \left( \frac{\theta + \phi}{2} \right) \cos \left( \frac{\theta - \phi}{2} \right) + \sin \left( \frac{\theta + \phi}{2} \right) \sin \left( \frac{\theta - \phi}{2} \right)$$

$$- \cos \left( \frac{\theta + \phi}{2} \right) \cos \left( \frac{\theta - \phi}{2} \right) + \sin \left( \frac{\theta + \phi}{2} \right) \sin \left( \frac{\theta - \phi}{2} \right)$$

$$= 2 \sin \left( \frac{\theta + \phi}{2} \right) \sin \left( \frac{\theta - \phi}{2} \right)$$
and \( \cos(\theta) = \cos \left( \frac{\theta}{2} + \frac{\theta}{2} \right) = \cos^2 \left( \frac{\theta}{2} \right) - \sin^2 \left( \frac{\theta}{2} \right) = 1 - 2 \sin^2 \left( \frac{\theta}{2} \right) \). Assuming that we have \( \sin \left( \frac{\theta}{2} \right) \geq 0 \) (for example, if \( 0 \leq \theta \leq 2\pi \)), we obtain \( \sin \left( \frac{\theta}{2} \right) = \sqrt{\frac{1 - \cos(\theta)}{2}} \).

In a spherical triangle \( ABC \), the angles are less than \( \pi \), so the second identity from the previous paragraph establishes that \( \sin \left( \frac{\alpha}{2} \right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \). The spherical cosine law asserts that \( \cos(a) = \cos(b) \cos(c) + \sin(b) \sin(c) \cos(\alpha) \) or \( \cos(\alpha) = \frac{\cos(a) - \cos(b) \cos(c)}{\sin(b) \sin(c)} \), so we obtain

\[
\sin \left( \frac{\alpha}{2} \right) = \sqrt{\frac{\sin(b) \sin(c) - \cos(a) + \cos(b) \cos(c)}{2 \sin(b) \sin(c)}}.
\]

The first identity from the previous paragraph and the addition formula for cosine show that

\[
2 \sin(s - b) \sin(s - c) = 2 \sin \left( \frac{a - (b - c)}{2} \right) \sin \left( \frac{a + (b - c)}{2} \right) = \cos(b - c) - \cos(a) = \cos(b) \cos(c) + \sin(b) \sin(c) - \cos(a),
\]

which implies that \( \sin \left( \frac{\alpha}{2} \right) = \sqrt{\frac{\sin(s - b) \sin(s - c)}{\sin(b) \sin(c)}} \). Finally, applying this half-angle formula to the polar triangle associated to \( ABC \), we obtain

\[
\sin \left( \frac{\pi}{2} - \frac{\alpha}{2} \right) = \sqrt{\frac{\sin \left( \frac{\pi}{2} - (\sigma + \beta) \right) \sin \left( \frac{\pi}{2} - (\sigma + \gamma) \right)}{\sin(\pi - \beta) \sin(\pi - \gamma)}}
\]

Since \( \sin \left( \frac{\pi}{2} - \theta \right) = \cos(\theta) \) and \( \sin(\pi - \theta) = \sin(\theta) \), we have \( \cos \left( \frac{\alpha}{2} \right) = \sqrt{\frac{\cos(\sigma + \beta) \cos(\sigma + \gamma)}{\sin(\beta) \sin(\gamma)}} \). \( \square \)