1. Prove that any rigid motion is a product of at most three reflections.

**Solution.** Consider a rigid motion \( \varphi \) and let \( A, B, C \) be non-collinear points.

If \( A \neq \varphi(A) \), then define \( \ell \) to be the perpendicular bisector of the segment \( \overline{A\varphi(A)} \), and set \( \operatorname{Ref}_\ell \) to be the reflection in the line \( \ell \). When \( A = \varphi(A) \), let \( \operatorname{Ref}_\ell \) denote the rigid motion given by the identity map. In either case, we have \( \operatorname{Ref}_\ell(A) = \varphi(A) \), and we set \( B' := \operatorname{Ref}_\ell(B) \) and \( C' := \operatorname{Ref}_\ell(C) \).

If \( B' \neq \varphi(B) \), then define \( m \) to be the perpendicular bisector of the segment \( \overline{B'\varphi(B)} \), and set \( \operatorname{Ref}_m \) to be the reflection in the line \( m \). By definition, we have \( \operatorname{Ref}_m(B') = \varphi(B) \). Since \( \varphi \) and \( \operatorname{Ref}_\ell \) are rigid motions, we also have \( \varphi(A)\varphi(B) \cong \overline{AB} \cong \operatorname{Ref}_\ell(A)\operatorname{Ref}_\ell(B) = \varphi(A)\varphi(B) \), which implies that \( \varphi(A) \) lies on \( m \) and \( \operatorname{Ref}_m(\varphi(A)) = \varphi(A) \). On the other hand, if \( B' = \varphi(B) \), then let \( \operatorname{Ref}_m \) denote the rigid motion given by the identity map, so we still have \( \operatorname{Ref}_m(\varphi(A)) = \varphi(A) \) and \( \operatorname{Ref}_m(B') = \varphi(B) \). Set \( C'' := \operatorname{Ref}_m(C') \).

If \( C'' \neq \varphi(C) \), then define \( n \) to be the perpendicular bisector of the segment \( \overline{C''\varphi(C)} \), and set \( \operatorname{Ref}_n \) to be the reflection in the line \( n \). By definition, we have \( \operatorname{Ref}_n(C'') = \varphi(C) \). Since \( \varphi, \operatorname{Ref}_\ell, \) and \( \operatorname{Ref}_m \) are rigid motions, we also have

\[
\frac{\varphi(A)\varphi(C) \cong \overline{AC} \cong \operatorname{Ref}_\ell(A)\operatorname{Ref}_\ell(C)}{\varphi(B)\varphi(C) \cong \overline{BC} \cong \operatorname{Ref}_\ell(B)\operatorname{Ref}_\ell(C)} = \frac{\varphi(A)\varphi(C) \cong \overline{AC} \cong \operatorname{Ref}_m(\varphi(A))\operatorname{Ref}_m(C')}{\varphi(B)\varphi(C) \cong \overline{BC} \cong \operatorname{Ref}_m(\varphi(B))\operatorname{Ref}_m(C')} = \varphi(A)C''
\]

which implies that both \( \varphi(A) \) and \( \varphi(B) \) lie on \( n \). Hence, we have \( \operatorname{Ref}_n(\varphi(A)) = \varphi(A) \) and \( \operatorname{Ref}_n(\varphi(B)) = \varphi(B) \). On the other hand, if \( C'' = \varphi(C) \), then let \( \operatorname{Ref}_n \) denote the rigid motion given by the identity map, so we still have \( \operatorname{Ref}_n(\varphi(A)) = \varphi(A) \), \( \operatorname{Ref}_n(\varphi(B)) = \varphi(B) \), and \( \operatorname{Ref}_n(\varphi(C)) = \varphi(C) \).

By construction, the composite map \( \operatorname{Ref}_n \circ \operatorname{Ref}_m \circ \operatorname{Ref}_\ell \) is the product of at most three reflections such that

\[
(\operatorname{Ref}_n \circ \operatorname{Ref}_m \circ \operatorname{Ref}_\ell)(A) = \operatorname{Ref}_n(\operatorname{Ref}_m(\operatorname{Ref}_\ell(A))) = \operatorname{Ref}_n(\operatorname{Ref}_m(\varphi(A))) = \operatorname{Ref}_n(\varphi(A)) = \varphi(A)
\]

\[
(\operatorname{Ref}_n \circ \operatorname{Ref}_m \circ \operatorname{Ref}_\ell)(B) = \operatorname{Ref}_n(\operatorname{Ref}_m(\operatorname{Ref}_\ell(B))) = \operatorname{Ref}_n(\operatorname{Ref}_m(\varphi(B))) = \operatorname{Ref}_n(\varphi(B)) = \varphi(B)
\]

\[
(\operatorname{Ref}_n \circ \operatorname{Ref}_m \circ \operatorname{Ref}_\ell)(C) = \operatorname{Ref}_n(\operatorname{Ref}_m(\operatorname{Ref}_\ell(C))) = \operatorname{Ref}_n(\operatorname{Ref}_m(\varphi(C))) = \operatorname{Ref}_n(\varphi(C)) = \varphi(C)
\]

Since any rigid motion is completely determined by the images of any three non-collinear points, we conclude that \( \varphi = \operatorname{Ref}_n \circ \operatorname{Ref}_m \circ \operatorname{Ref}_\ell \). \(\square\)

2. Show that the sum of the interior angles for any triangle in the Poincaré Disk Model is less than two right angles.
Solution. In the solution we call triangles in the Poincaré disk model $D$-triangles for short. We first construct a $D$-triangle. Let $O$ denote the centre of the unit circle $S^1$ and let $A$ be another point in the interior of $S^1$. Choose the point $A'$ on the ray $\overrightarrow{OA}$ such that $[OA] \cdot [OA'] = 1$ (Hilb.C1) and consider the circle $\Gamma$ centred at the midpoint of $AA'$ with radius $OA$. It follows that $\Gamma$ is orthogonal to $S^1$ (Eucl.III.36), so the portion of $\Gamma$ lying inside $S^1$ is a $D$-line. Choose $B$ to be another point on the $D$-line defined by $\Gamma$ (Hilb.I2). Since any line through $O$ produces a $D$-line, it follows that $OAB$ is a $D$-triangle.

We next claim that the interior angles in the $D$-triangle $OAB$ sum to less than two right angles. By definition, the $D$-angle $\angle OAB$ is congruent to the angle $\alpha$ between the line $OA$ and the tangent line $\ell$ to $\Gamma$ at $A$ (and $\ell$ is perpendicular to $OA$ by construction). Similarly, the $D$-angle $\angle OBA$ is congruent to the angle $\beta$ between the line $OB$ and the tangent line $m$ to $\Gamma$ at $B$. Let $C$ denote the point inside $\Gamma$ where the line $OB$ meets the unit circle $S^1$. The line $m$ meets the side $\overline{OC}$ of the triangle $OAC$ at the point $C$. Since the points on the line $m$, except $B$, are outside the circle $\Gamma$, it follows that $m$ meets $\overline{OA}$ at a point $D$ (Hilb.B4). The $D$-angle $\angle AOB$ is congruent to the angle $\gamma := \angle OAB$. If $\delta := \angle ODB$, then the sum $\beta + \gamma + \delta$ of the interior angles in the Euclidean triangle $OBD$ equals two right angles (Eucl.I.32). Now, let $E$ denote the point inside $\Gamma$ where the line $OA$ meets the unit circle $S^1$. The line $\ell$ meets the side $\overline{DE}$ of the triangle $DEB$ at the point $E$. Since the points on the line $\ell$, except $A$, are outside the circle $\Gamma$, it follows that $\ell$ meets $\overline{BD}$ at a point $F$ (Hilb.B4). Hence, the exterior angle $\delta$ to the triangle $DAF$ is greater than $\alpha$ (Eucl.I.16). Therefore, the sum $\alpha + \beta + \gamma$ of the interior angles in the $D$-triangle $OAB$ is less than two right angles.

Having constructed a $D$-triangle in which the sum of interior angles is less than two right angles, we conclude that the $D$-model is semi-hyperbolic and that every $D$-triangle has this property. □

3. The hyperbolic sine and hyperbolic cosine functions are defined as $\sinh(x) := \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) := \frac{1}{2}(e^x + e^{-x})$. Given a hyperbolic triangle with vertices $A$, $B$, $C$, side lengths $a$, $b$, $c$, and
interior angles $\alpha, \beta, \gamma$, the hyperbolic law of cosines asserts that
\[
\cosh(a) = \cosh(b) \cosh(c) - \sinh(c) \sinh(b) \cos(\alpha).
\]
(a) Verify that $\cosh^2(x) - \sinh^2(x) = 1$.
(b) Using the hyperbolic law of cosines, derive the hyperbolic law of sines which asserts that
\[
\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.
\]

Solution.
(a) We have
\[
\cosh^2(x) - \sinh^2(x) = \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2
= \frac{1}{4}(e^{2x} + 2e^x + e^{-2x} - e^{2x} + 2e^{-x} - e^{2x} - e^{-2x}) = 1.
\]
(b) Using the hyperbolic law of cosines and part (a), we obtain
\[
\frac{\sin^2(\alpha)}{\sinh^2(a)} = \frac{1 - \cos^2(\alpha)}{\sinh^2(a)}
= \frac{\sinh^2(b) \sinh^2(c) - \cosh^2(b) \cosh^2(c) + 2 \cosh(a) \cosh(b) \cosh(c) - \cosh^2(a)}{\sinh^2(a) \sinh^2(b) \sinh^2(c)}
= \frac{1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2 \cosh(a) \cosh(b) \cosh(c)}{\sinh^2(a) \sinh^2(b) \sinh^2(c)}.
\]
Hence, we obtain
\[
\frac{\sinh(a)}{\sin(\alpha)} = \sqrt{\frac{\sinh^2(a) \sinh^2(b) \sinh^2(c)}{1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2 \cosh(a) \cosh(b) \cosh(c)}}.
\]
because $a \geq 0$ implies that $\sinh(a) \geq 0$, and $0 \leq \alpha + \beta + \gamma < \pi$ implies that $\sin(\alpha) \geq 0$.
Since the expression on the right is fixed under permutations of $a, b, c$ and $\alpha, \beta, \gamma$, the expression on the left is also fixed under these permutations. Therefore, we conclude that
\[
\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.
\]