Performance Analysis for Nonuniform Signaling over Flat Rayleigh Channels[1]

Libo Zhong, Fady Alajaji, and Glen Takahara
Dept. of Mathematics and Statistics
Queen's University, Kingston, Ontario, Canada, K7L 3N6
Email: {libo,fady,takahara}@mast.queensu.ca

Abstract — **We investigate the error performance of a communication system where a nonuniform memoryless binary source is transmitted via M-ary phase-shift keying (PSK) or quadrature amplitude modulation (QAM) over Rayleigh fading channels and demodulated via maximum a posteriori (MAP) detection. Using recently derived upper and lower bounds, which are tight and can be efficiently computed, the system symbol error (P_s) and bit error rates (P_b) are evaluated over a wide range of the signal-to-noise ratio (SNR).**

I. Introduction

The performance analysis of digital communication systems for fading channels has been an area of long-time interest [3, 5, 12, 13]. Specifically, important efforts have been devoted to the determination of the error rate for the transmission of data sources over noisy channels impaired by Rayleigh fading. In practice, many data sources (e.g., practical image and speech signals) are non-uniform; thus they contain amounts of natural redundancy. Even after data compression, a certain amount of residual redundancy is still exhibited due to the sub-optimality of the compression scheme [2, 4]. Characterization of the embedded residual redundancy can be achieved by modeling the bit stream as an independent and identically distributed (i.i.d.) nonuniform (Bernoulli) process or as a Markov process [1, 2, 4]. For non-uniform signaling, it is optimal to utilize the MAP decoder at the receiver, which minimizes the symbol error rate of the communication system. MAP decoding performs better than the maximum likelihood (ML) decoding, as the source becomes more non-uniform [1, 4].

Non-uniformly distributed signaling over AWGN channels was investigated in [11]. To the best of our knowledge, no published work has appeared that considers performance analysis of non-uniformly distributed signaling over fading channels. Thus, we extend the results obtained in [11] by focusing our study on fading environments, where the complex envelope of the channel response is known to approach a zero-mean complex Gaussian process, i.e., the envelope amplitude is Rayleigh and the phase is uniformly distributed [12].

The objective of this work is to evaluate the error performance when nonuniform M-ary signals are transmitted over a Rayleigh fading channel. If signal s_u is transmitted, a symbol error occurs when the decoded symbol equals any s_i, where $s_i \neq s_u$. Therefore, the symbol error rate (P_s) under MAP decoding can be written in terms of a union of events:

$$\begin{aligned} P_s &= \sum_{u=1}^{N} P(\epsilon \mid s_u) P(s_u) \\ &= \sum_{u=1}^{N} P\Big(\bigcup_{i\neq u} \epsilon_{ui} \;\Big|\; s_u\Big) P(s_u), \end{aligned} \tag{1}$$

where $P(\epsilon \mid s_u)$ is the conditional probability of error given that s_u was sent, and ϵ_{ui} represents the event that s_i has a higher MAP metric than s_u. The problem arising with determining the probability of symbol error using this formula is that the probability of a union of events is often difficult to compute explicitly, since in general it requires taking into account all combinations of event intersections.

The next feasible way is to investigate upper and lower bounds that are tight enough for estimating the true value of the error probability. Several tight upper and lower bounds for the probability of a finite union of events $P\Big(\bigcup_{i=1}^{N} A_i\Big)$ were developed recently, such as a lower bound established in [10] (the KAT lower bound), a practical algorithmic stepwise lower bound [11] originating from Kounias [8], and a greedy algorithmic implementation [11] of an upper bound due to Hunter [7]. In this work, we apply the above bounds, which are only expressed in terms of the individual event probabilities $P(A_i)$ and the pairwise event probabilities $P(A_i \cap A_j)$, to estimate the performance of non-uniform signaling over Rayleigh fading channels.

The rest of this paper is organized as follows. The problem of examining the symbol and bit error probabilities of nonuniform signals transmitted over Rayleigh fading channels used in conjunction with M-PSK/QAM modulation and MAP decoding is investigated in Section II. Numerical results and discussion are provided in Section III.

II. Nonuniform Signaling over Flat Rayleigh Fading Channels

Consider a nonuniform i.i.d. binary source $\{X_i\}$ (with distribution $P\{X = 0\} = p$) which is grouped in blocks of $\log_2 M$ bits (we assume that M is a power of 2). Each block is subsequently M-PSK or M-QAM modulated with Gray mapping. Then, the M-ary modulated signal sequence is transmitted over a frequency-nonselective flat Rayleigh fading channel and is decoded via the optimal MAP criterion at the receiver. More specifically, if one of M signals $s_1, s_2, \cdots, s_M$ is sent, then the MAP decoder declares that s_i was sent if, for $j = 1, 2, \cdots, M$

[1]This work was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada and the Premier Research Excellence Award (PREA) of Ontario.

and $j \neq i$, the MAP metric of s_i is bigger than the metric of s_j; i.e.,

$$P(s_i \mid r) \geq P(s_j \mid r), \tag{2}$$

where

$$r = cs_u + n$$

is the received complex signal, c and n are complex Gaussian distributed with zero-mean and covariance matrices $\sigma^2 I_2$ and $(N_0/2)I_2$, respectively, where I_2 is the 2×2 identity matrix. We assume that c, s_u and n are pairwise-uncorrelated.

Symbol Error Rate (SER) The symbol error rate P_s is expressed in (1). To apply the bounds on (1), we need to determine the $P(\epsilon_{ui} \mid s_u)$ and $P(\epsilon_{ui} \cap \epsilon_{uj} \mid s_u)$ event probabilities. If the complex-valued Gaussian random variable c can be estimated from the received signal without error, we can derive the conditional individual and pairwise error probabilities given the channel fading and that s_u is sent:

$$\begin{aligned} & P(\epsilon_{ui} \mid c, s_u) \\ = \ & \Pr\Big\{P(r \mid s_i, c)P(s_i) \geq P(r \mid s_u, c)P(s_u)\Big\} \\ = \ & Q\left(\phi_{ui}(\alpha)\right), \end{aligned} \tag{3}$$

and

$$\begin{aligned} & P(\epsilon_{ui} \cap \epsilon_{uj} \mid c, s_u) \\ = \ & \Pr\Big\{P(r \mid s_i, c)P(s_i) \geq P(r \mid s_u, c)P(s_u), \\ & \qquad P(r \mid s_j, c)P(s_j) \geq P(r \mid s_u, c)P(s_u)\Big\} \\ = \ & \Psi\left(\rho_{uij},\ \phi_{ui}(\alpha),\ \phi_{uj}(\alpha)\right), \end{aligned} \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm, $\langle \cdot, \cdot \rangle$ denotes the usual dot product, $\alpha = \|c\|$,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-t^2/2)dt, \tag{5}$$

$$\rho_{uij} = \frac{\langle s_i - s_u, s_j - s_u \rangle}{\|s_i - s_u\| \cdot \|s_j - s_u\|}, \tag{6}$$

$$\begin{aligned} \Psi(\rho_{uij}, a, b) = & \frac{1}{2\pi\sqrt{1-\rho_{uij}^2}} \\ & \int_a^{\infty}\int_b^{\infty} \exp\left[-\frac{(x^2 - 2\rho_{uij}xy + y^2)}{2(1-\rho_{uij}^2)}\right]dxdy, \end{aligned} \tag{7}$$

$$\phi_{ui}(\alpha) = \frac{\alpha d_{ui}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P(s_u)/P(s_i)}{2\alpha d_{ui}}, \tag{8}$$

and

$$\phi_{uj}(\alpha) = \frac{\alpha d_{uj}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P(s_u)/P(s_j)}{2\alpha d_{uj}}, \tag{9}$$

where $d_{ui} = \|s_i - s_u\|$.

Notice that

$$P(\epsilon_{ui} \mid c, s_u) = P(\epsilon_{ui} \mid \alpha, s_u)$$

and

$$P(\epsilon_{ui} \cap \epsilon_{uj} \mid c, s_u) = P(\epsilon_{ui} \cap \epsilon_{uj} \mid \alpha, s_u).$$

Therefore,

$$\begin{aligned} P(\epsilon_{ui} \mid s_u) &= E_\alpha\Big[P(\epsilon_{ui} \mid \alpha, s_u)\Big] \\ &= \int_0^{\infty} Q\left(\phi_{ui}(\alpha)\right) \frac{\alpha}{\sigma^2} \exp\Big(-\frac{\alpha^2}{2\sigma^2}\Big) d\alpha. \end{aligned} \tag{10}$$

Integrating the RHS of (10) by parts, we can show that

$$\begin{aligned} & P(\epsilon_{ui} \mid s_u) \\ = & \begin{cases} \frac{1}{2}\Big(1 - \frac{1}{\tau_{ui}}\Big)\exp\left[-\frac{\omega_{ui}}{2}(1+\tau_{ui})\right], \\ \qquad\qquad \text{if } \ \omega_{ui} \geq 0, \\ 1 - \frac{1}{2}\Big(1 + \frac{1}{\tau_{ui}}\Big)\exp\left[-\frac{\omega_{ui}}{2}(1-\tau_{ui})\right], \\ \qquad\qquad \text{if } \ \omega_{ui} < 0, \end{cases} \end{aligned} \tag{11}$$

where $\omega_{ui} = \ln P(s_u)/P(s_i)$, and

$$\tau_{ui} = \sqrt{(\sigma^2 d_{ui}^2 + 2N_0)\,/\,(\sigma^2 d_{ui}^2)}.$$

Similarly,

$$\begin{aligned} P(\epsilon_{ui} \cap \epsilon_{uj} \mid s_u) &= E_\alpha\Big[P(\epsilon_{ui} \cap \epsilon_{uj} \mid \alpha, s_u)\Big] \\ &= E_\alpha\Big[\Psi\left(\rho_{uij},\ \phi_{ui}(\alpha),\ \phi_{uj}(\alpha)\right)\Big], \end{aligned} \tag{12}$$

which we determine numerically.

We hence can apply the KAT lower, stepwise lower and greedy upper bounds [11] on (1) to obtain two lower bounds and one upper bound on P_s in terms of $P(\epsilon_{ui} \mid s_u)$, $P(\epsilon_{ui} \cap \epsilon_{uj} \mid s_u)$ and $P(s_u)$.

Bit Error Rate (BER) In many cases, the bit error rate P_b is a more useful performance measure. Under the MAP decoding criterion, P_b can be written as

$$P_b = \sum_{u=1}^{M} P_b(u) P(s_u),$$

where

$$\begin{aligned} P_b(u) &= \frac{1}{\log_2 M} E(\ \sharp \ \texttt{of bit errors} \mid s_u) \\ &= \frac{1}{\log_2 M} \sum_{m=1}^{M} d_H(w_m, w_u) A_{m|u}, \end{aligned}$$

and

$$\begin{aligned} A_{m|u} &= P(s_m \ \texttt{is decoded} \mid s_u) \\ &= 1 - P\Big(\bigcup_{i \neq m} \Big\{P(s_i \mid r) \geq P(s_m \mid r)\Big\} \ \Big| \ s_u\Big) \\ &= 1 - P\Big(\bigcup_{i \neq m} \epsilon_{umi} \ \Big| \ s_u\Big), \end{aligned}$$

where $u = 1, \cdots, M$, w_m and w_u are the bit assignments for signals s_m and s_u, respectively, $d_H(w_m, w_u)$ is the Hamming distance between w_m and w_u, and ϵ_{umi} represents the event that symbol s_i has a higher metric than symbol s_m.

As in the case for the symbol error rate, $P(\epsilon_{umi} \mid c, s_u)$ and $P(\epsilon_{umi} \cap \epsilon_{umj} \mid c, s_u)$ can be expressed in terms of the $Q(\cdot)$ and $\Psi(\cdot)$ functions, respectively. More precisely, we obtain that

$$\begin{aligned} & P(\epsilon_{umi} \mid c, s_u) \\ &= \Pr\Big\{ P(r \mid s_i, c)P(s_i) \geq P(r \mid s_m, c)P(s_m) \,\Big|\, s_u \Big\} \\ &= Q\left(\phi_{umi}(\alpha)\right), \end{aligned} \tag{13}$$

and

$$\begin{aligned} & P(\epsilon_{umi} \cap \epsilon_{umj} \mid c, s_u) \\ &= \Pr\Big\{ P(r \mid s_i, c)P(s_i) \geq P(r \mid s_m, c)P(s_m), \\ &\qquad P(r \mid s_j, c)P(s_j) \geq P(r \mid s_m, c)P(s_m) \,\Big|\, s_u \Big\} \\ &= \Psi\left(\rho_{mij},\ \phi_{umi}(\alpha),\ \phi_{umj}(\alpha)\right), \end{aligned} \tag{14}$$

where

$$\rho_{mij} = \frac{\langle s_i - s_m, s_j - s_m \rangle}{\|s_i - s_m\| \cdot \|s_j - s_m\|}, \tag{15}$$

$$\phi_{umi}(\alpha) = \frac{\sqrt{2N_0}\omega_{mi}}{2\alpha d_{mi}} + \frac{\alpha d_{ui}^2}{\sqrt{2N_0} d_{mi}} - \frac{\alpha d_{um}^2}{\sqrt{2N_0} d_{mi}},$$

and

$$\phi_{umj}(\alpha) = \frac{\sqrt{2N_0}\omega_{mj}}{2\alpha d_{mj}} + \frac{\alpha d_{uj}^2}{\sqrt{2N_0} d_{mj}} - \frac{\alpha d_{um}^2}{\sqrt{2N_0} d_{mj}}.$$

Therefore,

$$\begin{aligned} P(\epsilon_{umi} \mid s_u) &= E_\alpha\Big[P(\epsilon_{umi} \mid \alpha, s_u)\Big] \\ &= E_\alpha\Big[Q(\phi_{umi}(\alpha))\Big]. \end{aligned} \tag{16}$$

Integrating the RHS of (16) by parts, we can show that

$$P(\epsilon_{umi} \mid s_u) = \begin{cases} \frac{1}{2}\exp\Big\{-\frac{\omega_{mi}}{2d_{mi}^2}\left[\left(d_{ui}^2 - d_{um}^2\right) + \nu_{umi}\right]\Big\} \\ \times\left(1 - \frac{\left(d_{ui}^2 - d_{um}^2\right)}{\nu_{umi}}\right), & \text{if } \omega_{mi} \geq 0, \\ \\ 1 - \frac{1}{2}\exp\Big\{\frac{\omega_{mi}}{2d_{mi}^2}\left[-\left(d_{ui}^2 - d_{um}^2\right) + \nu_{umi}\right]\Big\} \\ \times\left(1 + \frac{\left(d_{ui}^2 - d_{um}^2\right)}{\nu_{umi}}\right), & \text{if } \omega_{mi} < 0, \end{cases} \tag{17}$$

where

$$\nu_{umi} = \sqrt{[\sigma^2(d_{ui}^2 - d_{um}^2)^2 + (2N_0 d_{mi}^2)]/\sigma^2}.$$

Similarly,

$$\begin{aligned} & P(\epsilon_{umi} \cap \epsilon_{umj} \mid s_u) \\ &= E_\alpha\Big[P(\epsilon_{umi} \cap \epsilon_{umj} \mid \alpha, s_u)\Big] \\ &= E_\alpha\Big[\Psi\left(\rho_{mij},\ \phi_{umi}(\alpha),\ \phi_{umj}(\alpha)\right)\Big], \end{aligned} \tag{18}$$

which we compute numerically.

Applying the bounds to $P\Big(\bigcup_{i \neq m} \epsilon_{umi} \,\Big|\, s_u\Big)$ yields two upper bounds and one lower bound on the bit error rate P_b.

III. Numerical Results and Discussion

We apply the KAT, the stepwise, and the greedy bounds to estimate the *SER* P_s and the *BER* P_b of an uncoded communication system used in conjunction with 8,16-PSK/16-QAM modulations and MAP decoding over a Rayleigh fading channel. We consider $p = 0.5$ and 0.9 for the probability that an input source bit is zero. All results are illustrated in terms of signal-to-noise ratio (SNR) E_b/N_0, where E_b is the energy per information bit. The SNR ranges considered are all from SNR $= 0$ dB to SNR $= h$, with h ranging from 26 dB to 30 dB. To verify the accuracy of the bounds, we also provide simulation results for $p = 0.5$ and 0.9, which are obtained by averaging 1000 trials with 100,000 symbols each.

For the case $p = 0.5$ with M-PSK and M-QAM modulation, there exist good approximations for P_s [12] and P_b [14]. However, for 8,16-PSK and 16-QAM modulation we have found the lower and upper bounds for both P_s and P_b derived in Section II, based on the stepwise and greedy bounds, to coincide, and agree with the approximate and simulation results. The stepwise and greedy-based bounds for symbol errors show excellent accuracy for $p = 0.5$ (see Figs. 1–3). Notice that MAP estimation is equivalent to ML estimation for $p = 0.5$. However, it is illustrated that the performance for highly nonuniform sources is significantly improved by using MAP decoding.

For the case of nonuniform signaling, the exact or approximate symbol- or bit-error rates are not available to the best of our knowledge; hence bounds are very helpful. We plot the symbol and bit error rates in Figs. 1–3 for $p = 0.9$. It is shown that the bounds provide an excellent estimate of the error probabilities over the entire range of SNR values. The stepwise and the greedy bounds are particularly impressive as they agree with the simulation results even during very severe channel conditions. The performance of the KAT lower bound for the SER is weaker than that of the stepwise lower bound for SER; but it is more precise for $p = 0.9$ than for $p = 0.5$.

For the cases considered in this paper, all bounds were practical to compute. In all the SER and BER curves, for a given value of SNR, and including the time to calculate all the individual and pairwise event probabilities, the combined computing time ranged from 3s (for the SER with 8-PSK) to 62s (for the BER with 16-PSK), on a Sun Ultra-Sparc 60 computer running Unix. To compute the $\Psi(\cdot)$ function, we adopted the algorithm written in Fortran by Donnelly [6]. Gaussian Quadrature was used to compute the integration over the fading attenuation for the pairwise event probabilities.

References

[1] F. Alajaji, N. Phamdo, N. Farvardin, and T. Fuja, "Detection of binary Markov sources over channels with additive Markov noise," *IEEE Trans. Inform. Theory*, vol. 42, pp. 230–239, Jan. 1996.

[2] F. Alajaji, N. Phamdo, and T. Fuja, "Channel codes that exploit the residual redundancy in CELP-encoded speech," *IEEE Trans. Speech Audio Processing*, vol. 4, pp. 325–336, Sept. 1996.

[3] W. R. Bennet, M. Schwartz, and S. Stein, *Communication Systems and Techniques*. New York: McGraw-Hill, 1966.

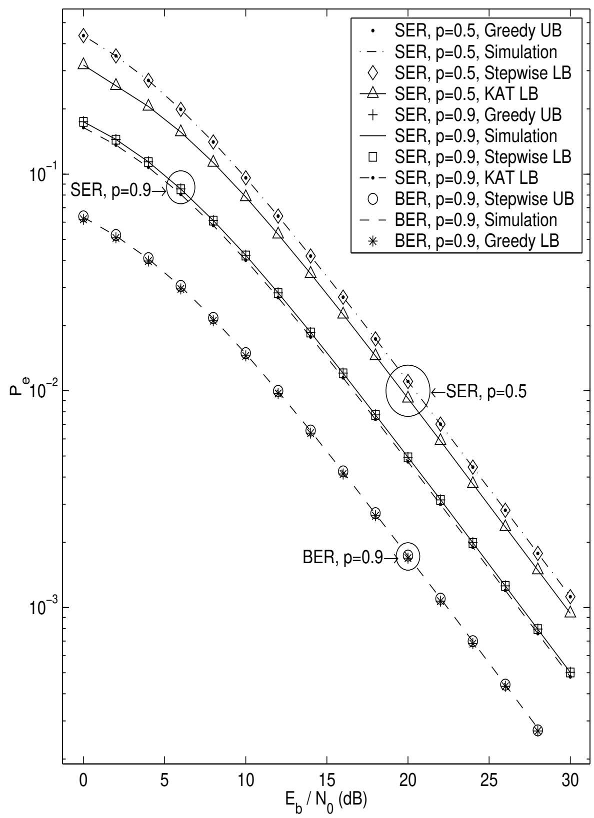

Fig. 1: SER P_s and BER P_b for 8-PSK.

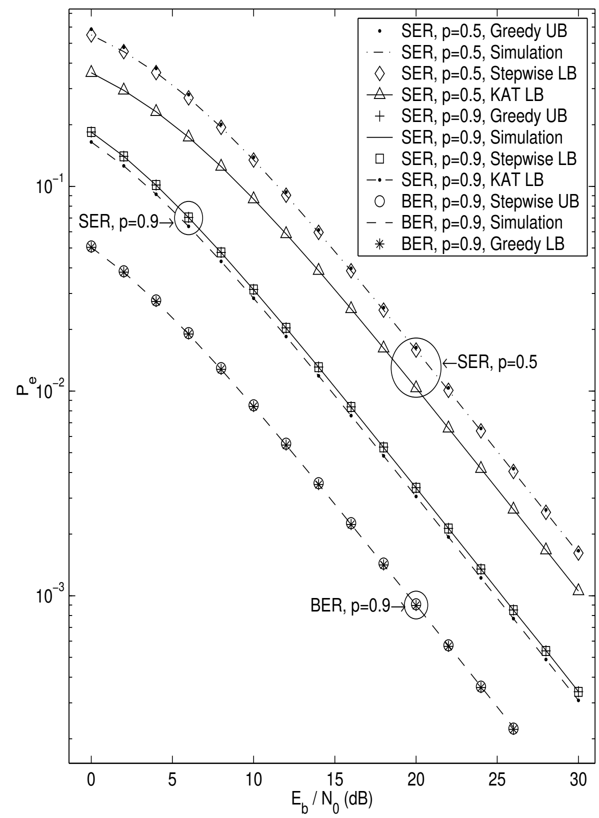

Fig. 3: SER P_s and BER P_b for 16-QAM.

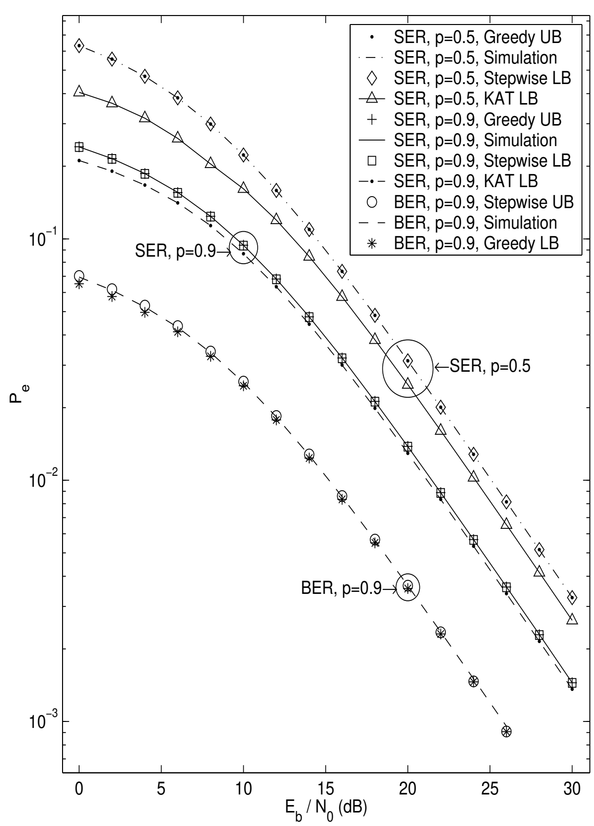

Fig. 2: SER P_s and BER P_b for 16-PSK.

[4] P. Burlina and F. Alajaji, "An error resilient scheme for image transmission over noisy channels with memory," *IEEE Trans. Image Processing*, vol. 7, pp. 593–600, Apr. 1998.

[5] X. Dong, N. C. Beaulieu, and P. H. Wittke, "Error probabilities of two-dimensional M-ary signaling in fading," *IEEE Trans. Commun.*, vol. 47, pp. 352–355, Mar. 1999.

[6] T. G. Donnelly, "Bivariate normal distribution," *Commun. ACM*, vol. 16, p. 636, 1973. Algorithm 462.

[7] D. Hunter, "An upper bound for the probability of a union," *J. Appl. Probab.*, vol. 13, pp. 597–603, 1976.

[8] E. G. Kounias, "Bounds on the probability of a union, with applications," *Ann. Math. Statist.*, vol. 39, no. 6, pp. 2154–2158, 1968.

[9] J. Kroll and N. Phamdo, "Analysis and design of trellis codes optimized for a binary symmetric Markov source with maximum a posteriori detection," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2977–2987, Nov. 1998.

[10] H. Kuai, F. Alajaji, and G. Takahara, "A lower bound on the probability of a finite union of events," *Discr. Math.*, vol. 215, pp. 147–158, Mar. 2000.

[11] ——, "Tight Error Bounds for Nonuniform Signaling over AWGN Channels," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2712–2718, Nov. 2000.

[12] J. G. Proakis, *Digital Communications*. New York: McGraw-Hill, 2000.

[13] M. K. Simon and M. S. Alouini, "A unified approach to the performance analysis of digital communication over generalized fading channels." *Proc. IEEE*, vol. 86, pp. 1860–1877, Sept. 1998.

[14] G. L. Stüber, *Principles of Mobile Communication*. Boston: Kluwer Academic, 1996.