Performance Bounds of Non-Uniform Signaling over AWGN Channels*

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Abstract — In this work, we present two bounds (one lower bound and one upper bound) on the probability of a union of a finite number of events. The bounds — which are expressed in terms of only the individual event probabilities and the pairwise event probabilities — are applied to examine the symbol error ($P_e$) and bit error ($P_b$) probabilities of an uncoded communication system used in conjunction with M-ary PSK/QAM modulations and maximum a posteriori (MAP) decoding over AWGN channels. It is shown that both bounds provide a good, and often excellent, estimate of the error probabilities over the entire range of the signal-to-noise ratio $E_b/N_0$. The upper bound on $P_e$ and the lower bound on $P_b$ are particularly impressive as they agree with the simulation results even during very severe channel conditions.

1 Introduction

We present two bounds (one lower bound and one upper bound) on the probability of the union of a finite family of events ($P(A_1 \cup \cdots \cup A_N)$) in terms of only the individual event probabilities $P(A_i)$’s and the pairwise event probabilities $P(A_i \cap A_j)$’s. It is demonstrated in [7] that the lower bound is always better than two similar lower bounds, one by de Caen [5] and the other by Dawson and Sankoff [4], that use the same information. The upper bound — which is expressed in terms of a weighted connected graph $G$ and its spanning tree — is based on a greedy algorithm which constructs the optimal spanning tree in $G$.

We then investigate the application of these bounds to the probabilities of symbol error and bit error of non-uniform coherent M-ary Phase-Shift Keying (PSK) and Quadrature Amplitude Modulation (QAM) signaling in the presence of additive white Gaussian noise (AWGN).

In previous related work [8], Séguin employed de Caen’s inequality to derive a lower bound on the probabilities of error for uniform M-ary signals derived from a binary linear code and sent over AWGN channels.

2 A Lower Bound on the Probability of a Union

Consider a finite family of events $A_1, A_2, \ldots, A_N$ in a finite\(^1\) probability space $(\Omega, P)$, where $N$ is a fixed positive integer. For each $x \in \Omega$, let $p(x) \triangleq P(\{x\})$, and let the degree of $x$ — denoted by $\text{deg}(x)$ — be the number of $A_i$’s that contain $x$. Define

$$B_i(k) \triangleq \{ x \in A_i : \text{deg}(x) = k \}$$

and

$$a_i(k) \triangleq P(B_i(k)),$$

where $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$. We obtain the following lemma.

Lemma 1 ([7])

$$P\left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{a_i(k)}{k}.$$

This brings us to the following result.

Theorem 1 ([7])

$$P\left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \left( \frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1 - \theta_i) P(A_i)} + \frac{\left(1 - \theta_i\right) P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right),$$

where

$$\theta_i \triangleq \frac{\beta_i}{\alpha_i} - \left[ \frac{\beta_i}{\alpha_i} \right],$$

$$\alpha_i \triangleq \sum_{k=1}^{N} a_i(k) = P(A_i),$$

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\(1\)For a general probability space, the problem can be directly reduced to the finite case since there are only finitely many Boolean atoms specified by the $A_i$’s [5].
In this section, we provide an algorithmic upper bound on the probability of a union. As in the case for the lower bound, this upper bound is expressed in terms of the individual and pairwise error probabilities.

**Theorem 2 ([6])** Let $A_1, A_2, \ldots, A_N$ be $N$ sets, where $N \geq 3$. Then

$$\Pr\left( \bigcup_{i=1}^{N} A_i \right) \leq \sum_{i=1}^{N} \Pr(\bar{A}_i) - \sum_{(i,j) \in T_0} \Pr(A_i \cap A_j),$$

where $T_0$ is any tree spanning the $N$ indices of the sets $A_1, A_2, \ldots, A_N$ and $(i,j)$ is an edge in $T_0$.

We next apply the Greedy Algorithm in [9] which constructs the optimal spanning tree $T_0$ such that the second term in the right hand side of (2) is maximized [9][Thm. 2.2].

**Greedy Algorithm:** Consider a (fully) connected graph $G$ with $N$ vertices and $\binom{N}{2}$ edges $(i,j)$ of weights $P(A_i \cap A_j)$. Construct a set of edges $T_0$ as follows.

**Step 0:** $T_0 = \emptyset$.

**Step 1:** Add to $T_0$ the edge with maximum weight.

**Step 2:** From the remaining edges, add to $T_0$ the edge with maximum weight subject to the constraint that $T_0$ remain cycle-free.

**Step 3:** Repeat Step 2 until $T_0$ contains $N-1$ edges.

### 4 Non-Uniform Signaling over AWGN Channels

#### 4.1 Problem Formulation

We apply the bounds to estimate the symbol error probability ($P_s$) and the bit error probability ($P_b$) of non-uniform $M$-PSK or $M$-QAM modulated additive white Gaussian noise (AWGN) channels. The problem formulation is as follows.

We consider a non-uniform\footnote{The justification for the non-uniformity assumption of the source is as follows. In many practical image and speech compression techniques, after some transformation, the transform coefficients are turned into bit streams (binary source). Due to the sub- or $M$-QAM modulation (with Gray mapping) over an AWGN channel with single-sided power spectral density $N_0$. The source stream is grouped in blocks of $\log_2 M$ bits which are each subsequently mapped to a modulation signal for transmission over the channel.} independent and identically distributed (i.i.d.) binary source $\{X_i\}$, with distribution $P(X = 0) = p$, that is transmitted via $M$-PSK or $M$-QAM channels.

The source is as follows. In many practical image and speech compression techniques, after some transformation, the transform coefficients are turned into bit streams (binary source). Due to the sub-
4.1.2 Bit Error Rate: Under the MAP decoding criterion, the bit error probability $P_b$ can be written as

$$P_b = \sum_{j=1}^{M} P_b(j) P(s_j),$$

where

$$P_b(j) = \frac{1}{\log_2 M} E(\text{# of bit errors}| s_j \text{ is sent})$$

$$= \frac{1}{\log_2 M} \sum_{m=1}^{M} d(c_m, c_j) A_{m/j},$$

and

$$A_{m/j} = P(s_m \text{ is decoded}| s_j \text{ is sent})$$

$$= 1 - P \left( \bigcup_{i \neq m} \{ P(s_i|r) > P(s_m|r) | s_j \text{ is sent} \} \right)$$

$$= 1 - P \left( \bigcup_{i \neq m} \epsilon_{imj} \right),$$

where $j = 1, \ldots, M$, $c_m$ and $c_j$ are the (Gray coded) bit assignments for signals $s_m$ and $s_j$, respectively, $d(c_m, c_j)$ is the Hamming distance between $c_m$ and $c_j$, and $\epsilon_{imj}$ represents the event that symbol $s_i$ has a higher metric than symbol $s_m$ given that symbol $s_j$ was sent. As in the case for the symbol error probability, we can derive $P(\epsilon_{imj})$ and $P(\epsilon_{imj} \cap \epsilon_{kmj})$ in terms of the $Q(\cdot)$ and $\Psi(\cdot)$ functions. Finally, applying Theorems 1 and 2 (with the Greedy Algorithm) to $P(\bigcup_{i \neq m} \epsilon_{im})$ in (5) yields an upper bound and a lower bound to the bit error probability $P_b$, respectively.

5 Numerical Results

The computation of the two bounds to the probability of symbol error $P_s$ and bit error $P_b$ are performed for the 8-PSK, 16-PSK, 32-PSK and 16-QAM modulation systems. The results for $p = 0.5$ and $0.9$ are displayed in terms of the SNR $E_b/N_0$, where $E_b$ is the energy per information bit, in Figures 1-8. For the sake of comparison, we provide the actual simulation results for each system. We also provide the union upper bound in the $P_s$ plots. Note that when $p = 0.5$, MAP decoding reduces to maximum likelihood (ML) decoding. The chosen values of $E_b/N_0$ correspond to a very noisy channel environment (e.g. $E_b/N_0 \leq 6$ dB for the 8-PSK system).

We observe from the $P_s$ plots in Figures 1-5 that the lower bound converges to the upper bound rapidly, and that both bounds are very good over the entire considered range of $E_b/N_0$. More importantly, the upper bound is extremely close to the simulation results for all values of $E_b/N_0$, thus providing a very accurate estimate of the exact symbol error probability.

Finally, we remark from Figures 6-8 that in the case of the bit error rate bounds, while the upper bound is not always good (especially for highly non-uniform sources), the lower bound completely coincides with the simulation results.

6 Conclusion

A lower bound and an algorithmic upper bound on the probability of a finite union of events were provided in terms of only the individual and pairwise event probabilities. The benefits of these bounds were illustrated via the estimation of the symbol and bit error probabilities of non-uniform M-PSK/QAM signaling over very noisy AWGN communication channels used in conjunction with MAP decoding. The upper bound on $P_s$ and the lower bound on $P_b$ provided an excellent estimate of the exact error probability. Future work may include the application of these bounds to Rayleigh fading channels and the analysis of channel coded communications systems.

References


Comparison of Bounds For Symbol Error Using 8-PSK Modulation, $p=0.5$

Figure 1: Bounds for $P_s$ using 8-PSK modulation and $p = 0.5$ (ML decoding).

Comparison of Bounds For Symbol Error Using 16-QAM Modulation, $p=0.5$

Figure 3: Bounds for $P_s$ using 16-QAM modulation and $p = 0.5$ (ML decoding).

Comparison of Bounds For Symbol Error Using 8-PSK Modulation, $p=0.9$

Figure 2: Bounds for $P_s$ using 8-PSK modulation and $p = 0.9$ (MAP decoding).

Comparison of Bounds For Symbol Error Using 16-QAM Modulation, $p=0.9$

Figure 4: Bounds for $P_s$ using 16-QAM modulation and $p = 0.9$ (MAP decoding).
Comparison of Bounds For Symbol Error Using 32-PSK Modulation, $p=0.9$

Figure 5: Bounds for $P_s$ using 32-PSK modulation and $p = 0.9$ (MAP decoding).

Comparison of Bounds For Bit Error Using 8-PSK Modulation, $p=0.9$

Figure 7: Bounds for $P_b$ using 8-PSK modulation and $p = 0.9$ (MAP decoding).

Comparison of Bounds For Bit Error Using 8-PSK Modulation, $p=0.5$

Figure 6: Bounds for $P_b$ using 8-PSK modulation and $p = 0.5$ (ML decoding).

Comparison of Bounds For Bit Error Using 16-PSK Modulation, $p=0.9$

Figure 8: Bounds for $P_b$ using 16-PSK modulation and $p = 0.9$ (MAP decoding).