Performance Bounds of Non-Uniform Signaling over AWGN Channels*

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Abstract — In this work, we present two bounds (one lower bound and one upper bound) on the probability of a union of a finite number of events. The bounds – which are expressed in terms of only the individual event probabilities and the pairwise event probabilities – are applied to examine the symbol error (P_s) and bit error (P_b) probabilities of an uncoded communication system used in conjunction with M-ary PSK/QAM modulations and maximum a posteriori (MAP) decoding over AWGN channels. It is shown that both bounds provide a good, and often excellent, estimate of the error probabilities over the entire range of the signal-to-noise ratio E_b/N_0. The upper bound on P_s and the lower bound on P_b are particularly impressive as they agree with the simulation results even during very severe channel conditions.

1 Introduction

We present two bounds (one lower bound and one upper bound) on the probability of the union of a finite family of events ($P(A_1 \cup \cdots \cup A_N)$) in terms of only the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s. It is demonstrated in [7] that the lower bound is always better than two similar lower bounds, one by de Caen [5] and the other by Dawson and Sankoff [4], that use the same information. The upper bound – which is expressed in terms of a weighted connected graph G and its spanning tree – is based on a greedy algorithm which constructs the optimal spanning tree in G.

We then investigate the application of these bounds to the probabilities of symbol error and bit error of non-uniform coherent M-ary Phase-Shift Keying (PSK) and Quadrature Amplitude Modulation (QAM) signaling in the presence of additive white Gaussian noise (AWGN).

In previous related work [8], Séguin employed de Caen's inequality to derive a lower bound on the probabilities of error for uniform M-ary signals derived from a binary linear code and sent over AWGN channels.

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2 A Lower Bound on the Probability of a Union

Consider a finite family of events $A_1, A_2, \ldots, A_N$ in a finite[1] probability space (Ω, P), where N is a fixed positive integer. For each $x \in \Omega$, let $p(x) \stackrel{\triangle}{=} P(\{x\})$, and let the degree of x – denoted by $deg(x)$ – be the number of A_i's that contain x. Define

$$B_i(k) \stackrel{\triangle}{=} \{x \in A_i : \deg(x) = k\}$$

and

$$a_i(k) \stackrel{\triangle}{=} P(B_i(k)),$$

where $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$. We obtain the following lemma.

Lemma 1 ([7])

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{a_i(k)}{k}.$$

This brings us to the following result.

Theorem 1 ([7])

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i) P(A_i)} + \frac{(1-\theta_i) P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right), \quad (1)$$

where

$$\theta_i \stackrel{\triangle}{=} \frac{\beta_i}{\alpha_i} - \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor,$$

$$\alpha_i \stackrel{\triangle}{=} \sum_{k=1}^{N} a_i(k) = P(A_i),$$

[1] For a general probability space, the problem can be directly reduced to the finite case since there are only finitely many Boolean atoms specified by the A_i's [5].

and

$$\beta_i \triangleq \sum_{k=1}^{N}(k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j).$$

In [7], we also demonstrate that the above lower bound is always sharper than two similar lower bounds, one by de Caen [5] and the other by Dawson and Sankoff [4].

3 An Upper Bound on the Probability of a Union

In this section, we provide an algorithmic upper bound on the probability of a union. As in the case for the lower bound, this upper bound is expressed in terms of the individual and pairwise error probabilities.

Theorem 2 ([6]) Let $A_1, A_2, \cdots, A_N$ be N sets, where $N \geq 3$. Then

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \sum_{(i,j)\in T_0} P(A_i \bigcap A_j), \qquad (2)$$

where T_0 is any tree spanning the N indices of the sets $A_1, A_2, \cdots, A_N$ and (i,j) is an edge in T_0.

We next apply the Greedy Algorithm in [9] which constructs the *optimal* spanning tree T_0 such that the second term in the right hand side of (2) is maximized [9][Thm. 2.2].

Greedy Algorithm: Consider a (fully) connected graph G with N vertices and $\left(\begin{array}{c} N \\ 2 \end{array}\right)$ edges (i,j) of weights $P(A_i \cap A_j)$. Construct a set of edges T_0 as follows.

Step 0: $T_0 = \emptyset$.

Step 1: Add to T_0 the edge with maximum weight.

Step 2: From the remaining edges, add to T_0 the edge with maximum weight subject to the constraint that T_0 remain cycle-free.

Step 3: Repeat Step 2 until T_0 contains $N-1$ edges.

4 Non-Uniform Signaling over AWGN Channels

4.1 Problem Formulation

We apply the bounds to estimate the symbol error probability (P_s) and the bit error probability (P_b) of non-uniform M-PSK or M-QAM modulated additive white Gaussian noise (AWGN) channels. The problem formulation is as follows.

We consider a *non-uniform*[2] independent and identically distributed (i.i.d.) binary source $\{X_i\}$, with distribution $P\{X=0\}=p$, that is transmitted via M-PSK or M-QAM modulation (with Gray mapping) over an AWGN channel with single-sided power spectral density N_0. The source stream is grouped in blocks of $\log_2 M$ bits which are each subsequently mapped to a modulation signal for transmission over the channel. At the receiver, optimal maximum a posteriori decoding (MAP) is performed in estimating the transmitted M-ary signal. More specifically, if one of M signals $s_1, s_2, \ldots, s_M$ is sent, then the MAP decoder declares that s_k was sent if, for $i = 1, 2, \ldots, M$ and $i \neq k$, the MAP metric of s_k is bigger than the metric of s_i; i.e.,

$$P\left(s_k|r\right) > P\left(s_i|r\right),$$

where r is the received signal.

4.1.1 Symbol Error Rate: The probability of symbol decoding error P_s is

$$P_s = \sum_{u=1}^{M} P(\epsilon|s_u)P(s_u) = \sum_{u=1}^{M} P\Big(\bigcup_{i\neq u}\epsilon_{ui}\Big)P(s_u), \qquad (3)$$

where $P(\epsilon|s_u)$ is the conditional probability of error given that s_u was sent, and ϵ_{ui} represents the event that s_i has a higher MAP metric than s_u given that s_u was sent. It can be shown that

$$P\left(\epsilon_{ui}\right) = Q\left(\frac{d_{ui}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_i\right)}{2d_{ui}}\right),$$

and

$$P\left(\epsilon_{ui}\cap\epsilon_{uj}\right) = \Psi\left(\rho_{ij}, \frac{d_{ui}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_i\right)}{2d_{ui}}, \right.$$
$$\left. \frac{d_{uj}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0}\ln P\left(s_u\right)/P\left(s_j\right)}{2d_{uj}}\right),$$

where

$$d_{ui} = ||s_i - s_u||,$$

$$\rho_{ij} = \frac{< s_i - s_u, s_j - s_u >}{||s_i - s_u|| \cdot ||s_j - s_u||},$$

$$Q\left(x\right) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} exp\left(-y^2/2\right)dy$$

$$\Psi\left(\rho_{ij}, a, b\right) = \frac{1}{2\pi\sqrt{1-\rho_{ij}^2}}\int_a^{\infty}\int_b^{\infty} e^{-\frac{\left(x^2-2\rho_{ij}xy+y^2\right)}{2\left(1-\rho_{ij}^2\right)}}\,dx\,dy,$$

and where $||\cdot||$ is the Euclidean norm and $<\cdot,\cdot>$ denotes the usual dot product.

If we apply Theorems 1 and 2 (along with the Greedy Algorithm) to $P(\bigcup_{i\neq u}\epsilon_{ui})$ and substitute in (3), we obtain a lower and an upper bound on P_s in terms of $P(\epsilon_{ui})$, $P(\epsilon_{ui}\cap\epsilon_{uj})$ and $P(s_u)$.

[2]The justification for the non-uniformity assumption of the source is as follows. In many practical image and speech compression techniques, after some transformation, the transform coefficients are turned into bit streams (binary source). Due to the suboptimality of the compression scheme, the bit stream often exhibits a certain amount of redundancy [2, 3, 10]. This embedded residual redundancy can be characterized by modeling the bitstream as an i.i.d. non-uniform process or as a Markov process [1, 2, 3].

4.1.2 Bit Error Rate: Under the MAP decoding criterion, the bit error probability P_b can be written as

$$P_b = \sum_{j=1}^{M} P_b(j) P(s_j),$$

where

$$\begin{aligned} P_b(j) &= \frac{1}{\log_2 M} E(\texttt{\# of bit errors} | s_j \texttt{ is sent}) \\ &= \frac{1}{\log_2 M} \sum_{m=1}^{M} d(c_m, c_j) A_{m/j}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} A_{m/j} &= P(s_m \texttt{ is decoded} | s_j \texttt{ is sent}) \\ &= 1 - P\Big(\bigcup_{i \neq m} \{P(s_i|r) > P(s_m|r) | s_j \texttt{ is sent}\}\Big) \\ &= 1 - P\Big(\bigcup_{i \neq m} \epsilon_{imj}\Big), \end{aligned} \tag{5}$$

where $j = 1, \ldots, M$, c_m and c_j are the (Gray coded) bit assignments for signals s_m and s_j, respectively, $d(c_m, c_j)$ is the Hamming distance between c_m and c_j, and ϵ_{imj} represents the event that symbol s_i has a higher metric than symbol s_m given that symbol s_j was sent. As in the case for the symbol error probability, we can derive $P(\epsilon_{imj})$ and $P(\epsilon_{imj} \cap \epsilon_{kmj})$ in terms of the $Q(\cdot)$ and $\Psi(\cdot)$ functions. Finally, applying Theorems 1 and 2 (with the Greedy Algorithm) to $P(\bigcup_{i \neq m} \epsilon_{imj})$ in (5) yields an upper bound and a lower bound to the bit error probability P_b, respectively.

5 Numerical Results

The computation of the two bounds to the probability of symbol error P_s and bit error P_b are performed for the 8-PSK, 16-PSK, 32-PSK and 16-QAM modulation systems. The results for $p = 0.5$ and 0.9 are displayed in terms of the SNR E_b/N_0, where E_b is the energy per information bit, in Figures 1-8. For the sake of comparison, we provide the actual simulation results for each system. We also provide the union upper bound in the P_s plots. Note that when $p = 0.5$, MAP decoding reduces to maximum likelihood (ML) decoding. The chosen values of E_b/N_0 correspond to a very noisy channel environment (e.g. $E_b/N_0 \leq 6$ dB for the 8-PSK system).

We observe from the P_s plots in Figures 1-5 that the lower bound converges to the upper bound rapidly, and that both bounds are very good over the entire considered range of E_b/N_0. More importantly, the upper bound is extremely close to the simulation results for all values of E_b/N_0, thus providing a very accurate estimate of the exact symbol error probability.

Finally, we remark from Figures 6-8 that in the case of the bit error rate bounds, while the upper bound is not always good (specially for highly non-uniform sources), the lower bound completely coincides with the simulation results.

6 Conclusion

A lower bound and an algorithmic upper bound on the probability of a finite union of events were provided in terms of only the individual and pairwise event probabilities. The benefits of these bounds were illustrated via the estimation of the symbol and bit error probabilities of non-uniform M-PSK/QAM signaling over very noisy AWGN communication channels used in conjunction with MAP decoding. The upper bound on P_s and the lower bound on P_b provided an excellent estimate of the exact error probability. Future work may include the application of these bounds to Rayleigh fading channels and the analysis of channel coded communications systems.

References

[1] F. Alajaji, N. Phamdo, N. Farvardin and T. Fuja, "Detection of Binary Markov Sources over Channels with Additive Markov Noise," *IEEE Trans. Inform. Theory*, Vol. 42, No. 1, pp. 230–239, January. 1996.

[2] F. Alajaji, N. Phamdo and T. Fuja, "Channel Codes That Exploit the Residual Redundancy in CELP-Encoded Speech," *IEEE Trans. Speech and Audio Processing*, Vol. 4, No. 5, pp. 325-336, September 1996.

[3] P. Burlina and F. Alajaji, "An Error Resilient Scheme for Image Transmission Over Noisy Channels with Memory," *IEEE Trans. Image Processing*, Vol. 7, No. 4, pp. 593-600, April 1998.

[4] D. A. Dawson and D. Sankoff, "An Inequality for Probabilities," *Proc. Amer. Math. Soc.*, vol. 18, pp. 504-507, 1967.

[5] D. de Caen, "A Lower Bound on the Probability of a Union," *Discrete Mathematics*, Vol. 169, pp. 217-220, 1997.

[6] J. Galambos and I. Simonelli, *Bonferroni-type Inequalities with Applications*, Springer, 1996.

[7] H. Kuai, F. Alajaji and G. Takahara, "A Lower Bound on the Probability of a Finite Union of Events," submitted for publication to *Discrete Mathematics*, February 1998.

[8] G. E. Séguin, "A Lower Bound on the Error Probability for Signals in White Gaussian Noise," *IEEE Trans. Inform. Theory*, Vol. 44, No. 7, pp. 3168-3175, November 1998.

[9] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics,* Cambridge, 1992

[10] W. Xu, J. Hagenauer and J. Hollmann, "Joint Source-Channel Decoding Using the Residual Redundancy in Compressed Images," *Proc. International Conference on Communications*, Dallas, TX, June 1996.

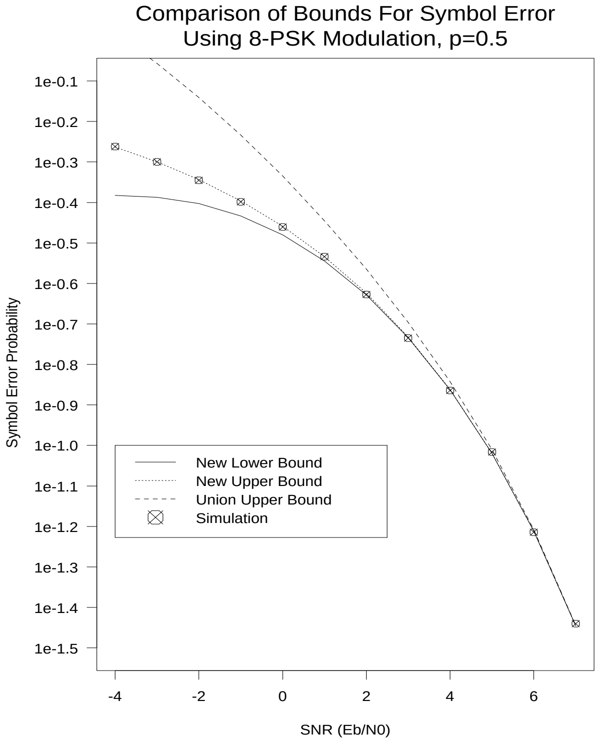

Figure 1: Bounds for P_s using 8-PSK modulation and $p = 0.5$ (ML decoding).

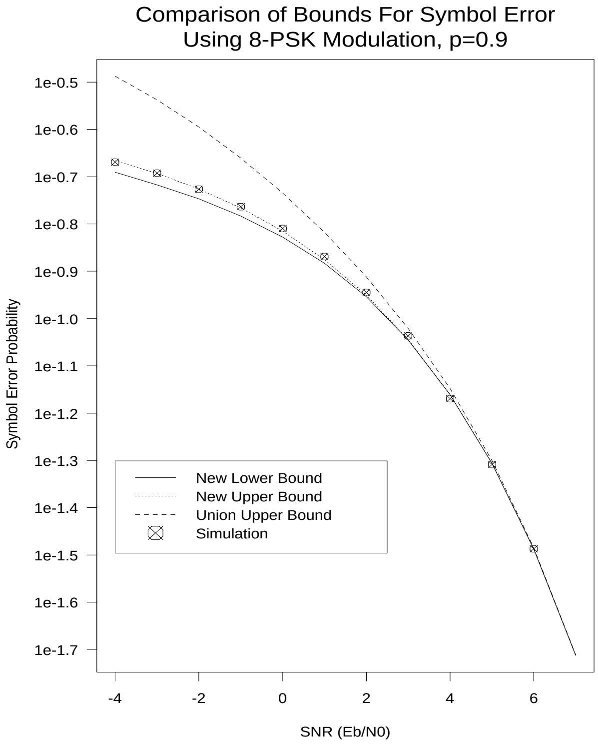

Figure 2: Bounds for P_s using 8-PSK modulation and $p = 0.9$ (MAP decoding).

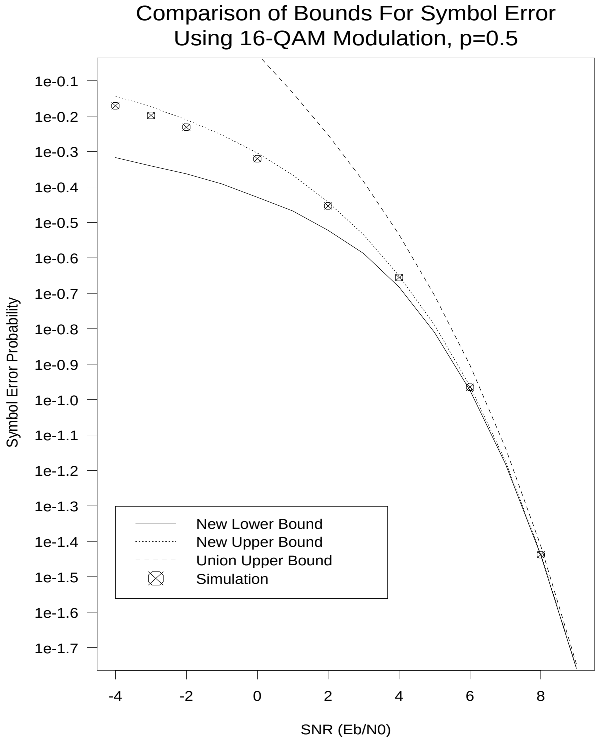

Figure 3: Bounds for P_s using 16-QAM modulation and $p = 0.5$ (ML decoding).

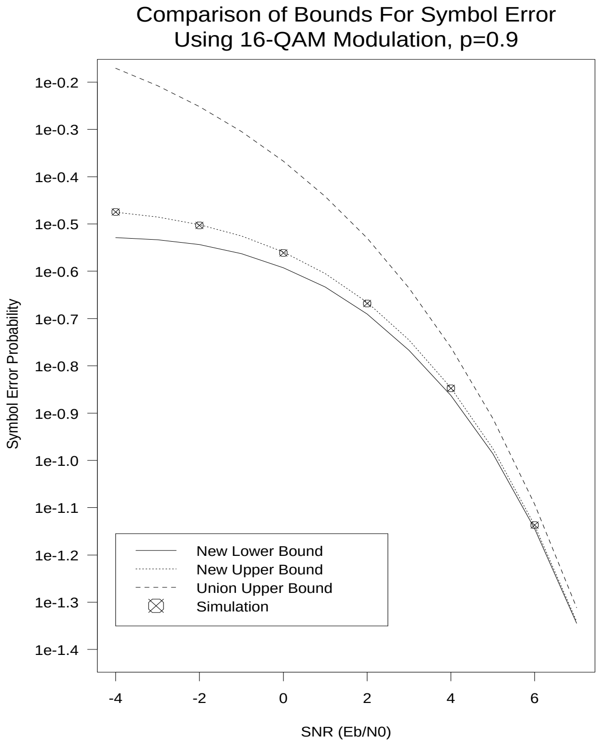

Figure 4: Bounds for P_s using 16-QAM modulation and $p = 0.9$ (MAP decoding).

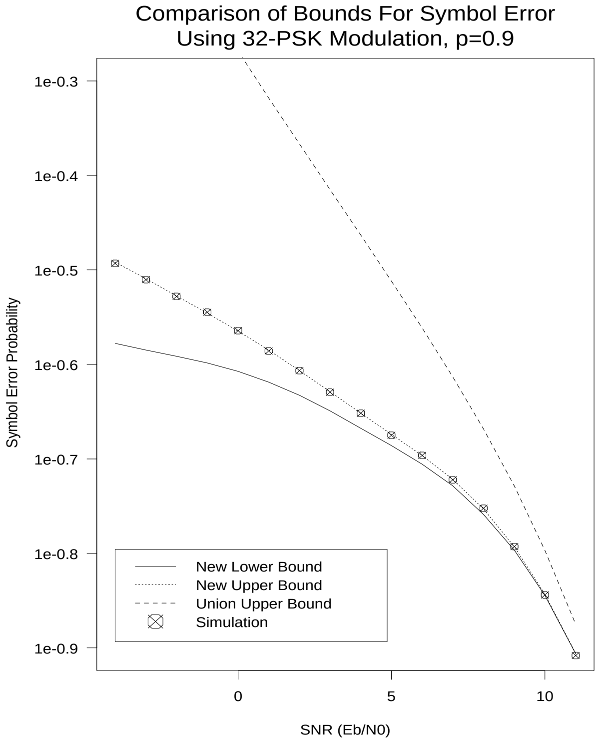

Figure 5: Bounds for P_s using 32-PSK modulation and $p = 0.9$ (MAP decoding).

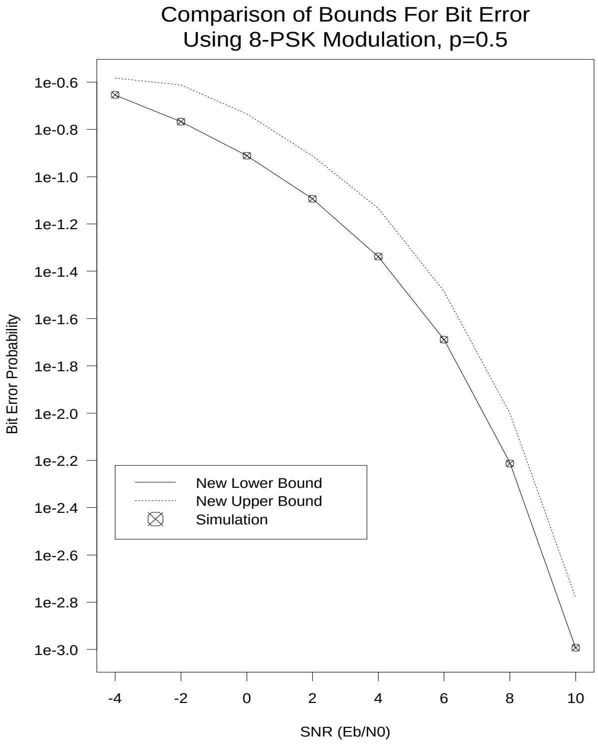

Figure 6: Bounds for P_b using 8-PSK modulation and $p = 0.5$ (ML decoding).

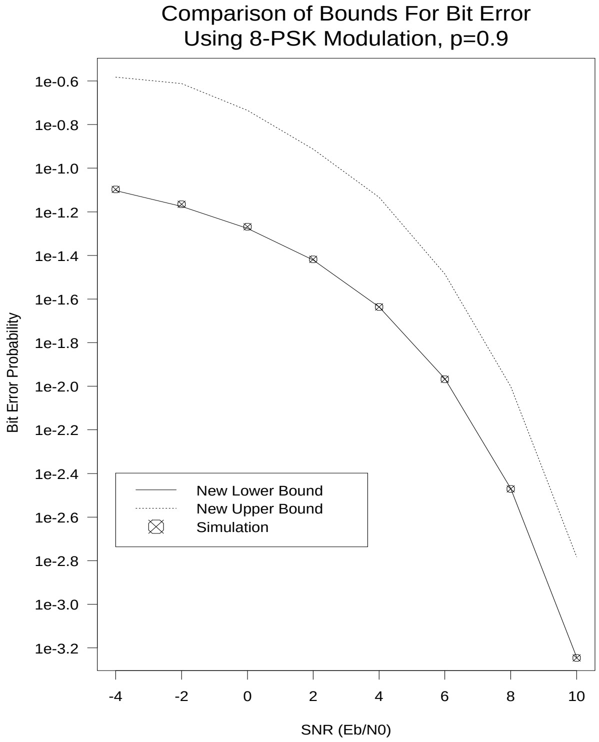

Figure 7: Bounds for P_b using 8-PSK modulation and $p = 0.9$ (MAP decoding).

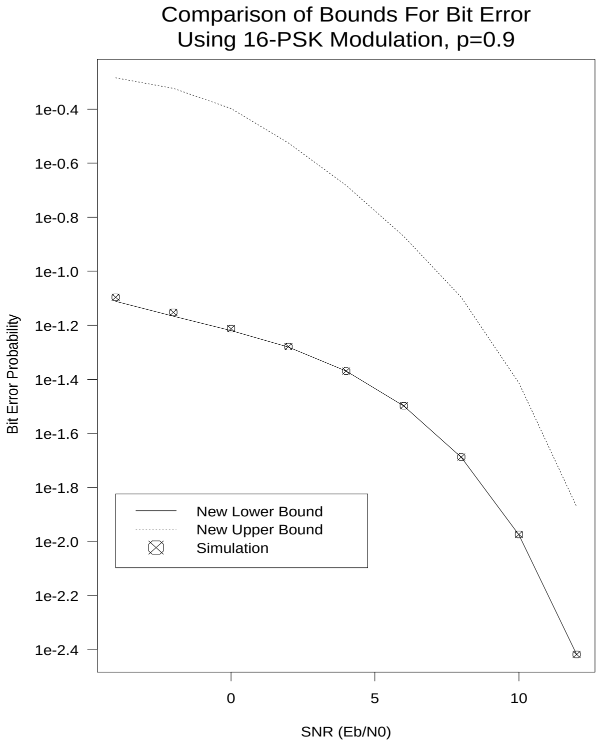

Figure 8: Bounds for P_b using 16-PSK modulation and $p = 0.9$ (MAP decoding).