On the Poor-Verdú Conjecture for the Reliability Function of Channels with Memory

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Abstract — **In a previous work, Poor and Verdú established an upper bound to the reliability function of arbitrary single-user discrete-time channels with memory. They also conjectured that their bound is tight for all coding rates. In this work, we demonstrate via a counterexample involving memoryless binary erasure channels that the Poor-Verdú upper bound is, unfortunately, not tight at low rates. We also examine possible improvements to this bound.**

I. Introduction

Consider an arbitrary input process $\mathbf{X}$ defined by a sequence of finite-dimensional distributions [4] $\mathbf{X} \triangleq \left\{X^n = \left(X_1^{(n)}, \cdots, X_n^{(n)}\right)\right\}_{n=1}^{\infty}$. Let $\mathbf{Y} \triangleq \left\{Y^n = \left(Y_1^{(n)}, \cdots, Y_n^{(n)}\right)\right\}_{n=1}^{\infty}$ be the corresponding output process induced by $\mathbf{X}$ via the channel $\mathbf{W} \triangleq \{W^n = P_{Y^n|X^n} : \mathcal{X}^n \to \mathcal{Y}^n\}_{n=1}^{\infty}$ which is an arbitrary sequence of n-dimensional conditional distributions from $\mathcal{X}^n$ to $\mathcal{Y}^n$, where $\mathcal{X}$ and $\mathcal{Y}$ are the input and output alphabets, respectively. We assume throughout that $\mathcal{X}$ is finite and that $\mathcal{Y}$ is arbitrary.

In [3], Poor and Verdú established an upper bound to the reliability function $E^*(R)$ of $\mathbf{W}$. They then conjectured that this bound is tight for all code rates. However, no known proof could substantiate this conjecture. In this work, we demonstrate via a counterexample that their original upper bound formula is not necessarily tight at low rates. A possible improvement to this bound is then addressed.

II. Preliminaries

For any $R > 0$, define the channel reliability function $E^*(R)$ for a channel $\mathbf{W}$ as the largest scalar $\beta > 0$ such that there exists a sequence of (n, M_n) codes with

$$\beta \le \liminf_{n\to\infty} -\frac{1}{n}\log_2 P_e(n, M_n)$$

and

$$R < \liminf_{n\to\infty} \frac{1}{n}\log_2 M_n,$$

where $P_e(n, M_n)$ is the code average error probability. For an input process $\mathbf{X}$ and channel $\mathbf{W}$, the large deviation spectrum of the channel is defined as

$$\pi_{\mathbf{X}}(R) \triangleq \liminf_{n\to\infty} -\frac{1}{n}\log_2 Pr\left[\frac{1}{n} i_{X^nW^n}(X^n;Y^n) \le R\right],$$

where $i_{X^nW^n}(X^n;Y^n) = \log_2(W^n(Y^n|X^n)/P_{Y^n}(Y^n))$ is the channel information density.

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Theorem 1 [3, 1] The channel reliability function satisfies

$$E^*(R) \le E_{PV}(R) \triangleq \sup_{\mathbf{X}\in\mathcal{Q}(R)} \pi_{\mathbf{X}}(R),$$

for any $R > 0$, where $\mathcal{Q}(R)$ is the set of all input processes $\mathbf{X}$ such that each X^n in $\mathbf{X}$ is uniformly distributed over its support $\mathcal{S}(X^n)$, and $R < \liminf_{n\to\infty} \frac{1}{n}\log_2|\mathcal{S}(X^n)|$.

III. Looseness of $E_{PV}(R)$ at low rates

Theorem 2 [1] For a binary erasure channel (BEC) with crossover probability $0 < \epsilon < 1$, the Poor-Verdú bound $E_{PV}(R)$ satisfies

$$E_{PV}(R) > E^*(R)$$

for $0 < R < 1 - \sqrt{\epsilon}$.

The key idea is to observe that for an input $\tilde{\mathbf{X}}$ with $\tilde{X}^n$ uniformly distributed over $\{0,1\}^n$, $\pi_{\tilde{\mathbf{X}}}(R)$ coincides with the space partitioning upper bound to $E^*(R)$, which is itself loose at low rates [2].

We can further restrict the condition on the input process to yield that for any $\rho > 0$,

$$E^*(R) \le E_{PV}^{(\rho)}(R) \triangleq \sup_{\mathbf{X}\in\mathcal{P}(R,\rho)} \pi_{\mathbf{X}}(R),$$

where $\mathcal{P}(R,\rho)$ is the set of all input processes $\mathbf{X}$ such that each X^n in $\mathbf{X}$ is uniformly distributed over its support $\mathcal{S}(X^n)$, and $R < \liminf_{n\to\infty} \frac{1}{n}\log_2|\mathcal{S}(X^n)| < R+\rho$. Although $E_{PV}^{(\rho)}(R)$ is an improvement of $E_{PV}(R)$, it is also unfortunately loose at rates close to zero. We have the following result.

Theorem 3 [1] Consider a BEC with crossover probability ϵ, and fix $\rho > 0$. The following holds.

$$\lim_{R\downarrow 0} E_{PV}^{\rho}(R) > \lim_{R\downarrow 0} E^*(R).$$

Remark: From the proofs of the Poor-Verdú upper bound in [3] and Theorem 1 in [1], the best upper bound that can be readily obtained is:

$$E^*(R) \le \inf_{\rho>0} E_{PV}^{(\rho)}(R).$$

Investigating the tightness of this bound at low rates for the BEC or other channels is an interesting future work.

References

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