Csiszár's Forward Cutoff Rate for Testing Between two Arbitrary Sources[1]

Fady Alajaji
Dept. of Math. & Stats.
Queen's University
Kingston, Ontario
Canada K7L 3N6
fady@mast.queensu.ca

Po-Ning Chen
Dept. of Comm. Engineering
Engineering Building 4
National Chiao Tung University
Hsin Chu, Taiwan 30050, R.O.C.
poning@cc.nctu.edu.tw

Ziad Rached
Dept. of Math. & Stats.
Queen's University
Kingston, Ontario
Canada K7L 3N6
rachedz@mast.queensu.ca

Abstract — **The Csiszár forward β-cutoff rate ($\beta < 0$) for hypothesis testing is defined as the largest rate $R_0 \geq 0$ such that for all rates $0 < E < R_0$, the smallest probability of type 1 error of sample size-n tests with probability of type 2 error $\leq e^{-nE}$ is asymptotically vanishing as $e^{-n\beta(E-R_0)}$. It was shown by Csiszár that the forward β-cutoff rate for testing between a null hypothesis $\mathbf{X}$ against an alternative hypothesis $\bar{\mathbf{X}}$ based on independent and identically distributed samples, is given by Rényi's α-divergence $D_\alpha(X\|\bar{X})$, where $\alpha = 1/(1-\beta)$.**

In this work, we show that the forward β-cutoff rate for the general hypothesis testing problem is given by the lim inf α-divergence rate. The result holds for an arbitrary abstract alphabet (not necessarily countable).

I. Introduction

In [2], Csiszár establishes the concept of forward β-cutoff rate for the hypothesis testing problem based on independent and identically (i.i.d.) observations. He then demonstrates that the forward β-cutoff rate is given by $D_{1/(1-\beta)}(X\|\bar{X})$, where $D_\alpha(X\|\bar{X})$ denotes the Rényi [4] α-divergence, $\alpha > 0$, $\alpha \neq 1$. This result provides a new operational significance for the α-divergence.

In this work, we extend Csiszár's result [2] by investigating the forward β-cutoff rate for the hypothesis testing between two arbitrary (not necessarily stationary, ergodic, etc.) sources with a general alphabet. We demonstrate that the liminf α-divergence rate provides the expression for the forward β-cutoff rate.

II. Preliminaries

Given two arbitrary sources $\mathbf{X}$ and $\bar{\mathbf{X}}$ taking values in the same source alphabet $\{\mathcal{X}^n\}_{n=1}^{\infty}$ [3], we may define the general hypothesis testing problem with $\mathbf{X}$ as the null hypothesis and $\bar{\mathbf{X}}$ as the alternative hypothesis. Let $\mathcal{A}_n$ be any subset of $\mathcal{X}^n$, $n = 1, 2, \ldots$ that we call the acceptance region of the hypothesis testing, and define $\mu_n \triangleq Pr\{X^n \notin \mathcal{A}_n\}$ and $\lambda_n \triangleq Pr\{\bar{X}^n \in \mathcal{A}_n\}$ where μ_n, λ_n are called type 1 error probability and type 2 error probability, respectively.

Definition 1 Fix $r > 0$. A rate E is called r-achievable if there exists an acceptance region $\mathcal{A}_n$ such that

$$\liminf_{n\to\infty} -\frac{1}{n}\log \mu_n \geq r \quad \text{and} \quad \liminf_{n\to\infty} -\frac{1}{n}\log \lambda_n \geq E.$$

[1]This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, the Premier's Research Excellence Award of Ontario, and the National Science Council of Taiwan.

Definition 2 The supremum of all r-achievable rates is denoted by $B_e(r|\mathbf{X}\|\bar{\mathbf{X}})$:

$$B_e(r|\mathbf{X}\|\bar{\mathbf{X}}) \triangleq \sup\{E \geq 0 : E \text{ is } r\textit{-achievable}\}.$$

The dual of this function is defined as:

$$D_e(E|\mathbf{X}\|\bar{\mathbf{X}}) \triangleq \sup\{r > 0 : E \text{ is } r\text{-achievable}\}.$$

III. Forward β-Cutoff Rate

Definition 3 Fix $\beta < 0$. $R_0 \geq 0$ is a forward β-achievable rate for the general hypothesis testing problem if

$$D_e(E|\mathbf{X}\|\bar{\mathbf{X}}) \geq \beta(E - R_0)$$

for every $E > 0$. The forward β-cutoff rate is defined as the supremum of all forward β-achievable rates, and is denoted by $R_0^{(f)}(\beta|\mathbf{X}\|\bar{\mathbf{X}})$. Our main result is the following.

Theorem 1 (Forward β-cutoff rate formula). Fix $\beta < 0$. For the general hypothesis testing problem,

$$R_0^{(f)}(\beta|\mathbf{X}\|\bar{\mathbf{X}}) = \liminf_{n\to\infty} \frac{1}{n} D_{\frac{1}{1-\beta}}(X^n\|\bar{X}^n),$$

where

$$D_\alpha(X^n\|\bar{X}^n) \triangleq \frac{1}{\alpha-1}\log\left(\sum_{x^n\in\mathcal{X}^n}[P_{X^n}(x^n)]^\alpha[P_{\bar{X}^n}(x^n)]^{1-\alpha}\right)$$

is the n-dimensional α-divergence.

The techniques used in our proof are a mixture of the techniques used in [1] for deriving the forward and reverse β-cutoff rates for source coding. However, some new techniques are also needed to obtain the result.

References

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