On the Optimistic Capacity of Arbitrary Channels*

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Abstract — A formula for the optimistic capacity of arbitrary channels is established. It is shown to equal the supremum over all input processes of the input-output zero-sup-information rate. A general expression for optimistic ε-capacity is also provided.

I. OVERVIEW

The conventional definition of channel capacity $C$ [1] requires the existence of reliable block codes for all sufficiently large blocklengths. Alternatively, if it is required that reliable codes exist for infinitely many blocklengths, a new, more optimistic definition of capacity is obtained [1]. This concept of optimistic capacity (denoted by $\overline{C}$) has recently been investigated by Verdú et al. for arbitrary single-user channels [1, 2]. More specifically, they provide an (additional) operational significance for the optimistic capacity by demonstrating that for a given channel, the classical statement of the source-channel separation theorem holds for every source if and only if $C = \overline{C}$ [2]. They also conjecture that a simple expression for $\overline{C}$ does not exist.

In this paper, we answer the latter point by demonstrating that $\overline{C}$ does indeed have a general formula. The key to showing this result is the application of the generalized sup-information rate introduced in [3] to the existing proofs by Verdú and Han [1] of the direct and converse parts of the conventional coding theorem. A general expression for the optimistic ε-capacity is also established.

II. ε-INF/SUP-INFORMATION RATES

Consider an input process $X^\Delta = \{X^n = (X^{(n)}, \ldots, X^{(n)})\}_{n=1}^\infty$ [1]. Denote by $Y^\Delta = \{Y^n = (Y^{(n)}, \ldots, Y^{(n)})\}_{n=1}^\infty$ the corresponding output processes induced by $X$ via the channel $W^\Delta = \{W^n = P^n_{X^nY^n} : X^n \rightarrow Y^n\}_{n=1}^\infty$. In [1, 4], Han and Verdú introduce the notions of inf/sup-information/entropy rates and illustrate the key role these measures play in proving general traditional source/channel coding theorems. The inf-information rate $I(X;Y)$ (resp. sup-information rate $\overline{I}(X;Y)$) between processes $X$ and $Y$ is defined in [4] as the liminf in probability (resp. limsup in prob.) of the sequence of normalized information densities $i_{X^nY^n}(X^n;Y^n)$. Definition 1 (ε-inf/sup-information rates [3]) is as follows:

The ε-inf-information rate $I_\epsilon(X;Y)$ and ε-sup-information rate $\overline{I}_\epsilon(X;Y)$ between $X$ and $Y$ are defined by

$\underline{I}_{\epsilon}(X;Y) = \liminf_{n \to \infty} \frac{1}{n} \min_{\delta > 0} \sup_{\delta \leq \epsilon} \{I_{\delta}(X;Y) \}

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$\overline{I}_{\epsilon}(X;Y) = \limsup_{n \to \infty} \frac{1}{n} \min_{\delta > 0} \sup_{\delta \leq \epsilon} \{I_{\delta}(X;Y) \}

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Note that Han and Verdú's inf/sup information rates are special cases of the above quantities: $I(X;Y) = I_0(X;Y)$ and $\overline{I}(X;Y) = \overline{I}_0(X;Y)$.

III. MAIN RESULTS

Definition 2 Given $0 < \epsilon < 1$, an $(n,M,\epsilon)$ code for channel $W$ has blocklength $n$, $M$ codewords, and average error probability not larger than $\epsilon$. $R \geq 0$ is an optimistic $\epsilon$-achievable rate if, for every $\delta > 0$, there exist, for infinitely many $n$, $(n,M,\epsilon)$ codes with rate $\frac{\log M}{n} > R - \delta$. The supremum of optimistic $\epsilon$-achievable rates is called the optimistic $\epsilon$-capacity, $\overline{C}_\epsilon$.

The optimistic channel capacity $\overline{C}$ is defined as the supremum of the rates that are optimistic $\epsilon$-achievable for all $0 < \epsilon < 1$.

Theorem 1 (Optimistic channel coding theorem)

$\overline{C} = \sup \underline{I}_\epsilon(X;Y)$.

Theorem 2 (Optimistic ε-capacity) For $0 < \epsilon < 1$, the optimistic ε-capacity $\overline{C}_\epsilon$ satisfies

$\sup \underline{I}_{\epsilon}(X;Y) \leq \overline{C}_\epsilon \leq \sup \overline{I}_{\epsilon}(X;Y)$.

Observations

- Recall that the general formula for the (pessimistic) capacity is $C = \sup X I(X;Y)$ [1]. It is known that for a DMC, $C = \overline{C}$. However, in general, $\overline{C} \geq C$ since $\overline{I}_0(X;Y) \geq I(X;Y)$ [3].

- A simple example of a channel for which $\overline{C} > C$ is as follows. Consider a nonstationary channel $W$ such that at odd time instances $n = 1, 3, \cdots, W_n$ is the transition distribution of a BSC with crossover probability 1/2; and at even time instances $n = 2, 4, 6, \cdots, W_n$ is the distribution of a BSC with crossover probability 1/4. Then $C = 0$ and $\overline{C} = 1 - h(1/4) > 0$.

- In [3], we further illustrate the application of the generalized information measures of [3] by proving an optimistic general source coding theorem.

REFERENCES


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