On the Computation of the Joint Source-Channel Error Exponent for Memoryless Systems[1]

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Abstract — **We study the analytical computation of Csiszár's [2] random-coding lower bound and sphere-packing upper bound for the lossless joint source-channel (JSC) error exponent, $E_J(Q,W)$, for a discrete memoryless source (DMS) Q and a discrete memoryless channel (DMC) W. We provide equivalent expressions for these bounds, which can be readily calculated for arbitrary (Q,W) pairs. We also establish explicit conditions under which the bounds coincide, thereby exactly determining $E_J(Q,W)$.**

I. Csiszár's Upper and Lower Bounds

Definition 1 A JSC code with blocklength n for a DMS with finite alphabet $\mathcal{S}$ and distribution Q, and a DMC with finite input alphabet $\mathcal{X}$, finite output alphabet $\mathcal{Y}$ and transition probability $W \triangleq P_{Y|X}$ is a pair of mappings $f_n : \mathcal{S}^n \longrightarrow \mathcal{X}^n$ and $\varphi_n : \mathcal{Y}^n \longrightarrow \mathcal{S}^n$. The code's average error probability is

$$P_e^{(n)}(Q,W) \triangleq \sum_{\{(s^n,y^n):\varphi_n(y^n)\neq s^n\}} Q(s^n)P_{Y|X}(y^n \mid f_n(s^n)).$$

Definition 2 The JSC error exponent $E_J(Q,W)$ for source $\{Q:\mathcal{S}\}$ and channel $\{W:\mathcal{X}\to\mathcal{Y}\}$ is defined as the largest number E for which there exists a sequence of JSC codes (f_n,φ_n) with $E \le \liminf_{n\to\infty} -\frac{1}{n}\log P_e^{(n)}(Q,W)$.

Proposition 1 [2] The JSC error exponent $E_J(Q,W)$ satisfies $\min_R[e(R,Q)+E_r(R,W)] \le E_J(Q,W) \le \min_R[e(R,Q)+E_{sp}(R,W)]$, where $e(R,Q)$ is the source error exponent, and $E_r(R,W)$ and $E_{sp}(R,W)$ are the random-coding lower bound and the sphere-packing upper bound for the channel error exponent, respectively.[2]

II. Main Results

Theorem 1 The JSC random-coding and sphere-packing bounds of Proposition 1 can be written as[3]

$$\max_{0\le\rho\le1}[E_o(\rho)-E_s(\rho)] \le E_J(Q,W) \le \max_{\rho\ge0}[E_o(\rho)-E_s(\rho)], \quad (1)$$

where $E_o(\rho)$ is Gallager's channel function

$$E_o(\rho) \triangleq \max_{P_X}\left[-\log\sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}}P_X(x)P_{Y|X}^{\frac{1}{1+\rho}}(y\mid x)\right)^{1+\rho}\right],$$

and $E_s(\rho)$ is Gallager's source function

$$E_s(\rho) \triangleq (1+\rho)\log\sum_{s\in\mathcal{S}}Q(s)^{\frac{1}{1+\rho}}.$$

From Theorem 1, we first note that Csiszar's JSC random coding lower bound, $\min_R[e(R,Q)+E_r(R,W)]$, is indeed identical to Gallager's lower bound established in [4, Problem 5.16] – as the latter bound is exactly the left-hand side bound in (1). We also remark that the minimizations in Proposition 1 are equivalent to more concrete maximizations of $E(\rho)\triangleq E_o(\rho)-E_s(\rho)$, which boil down to determining $E_o(\rho)$. Although $E_o(\rho)$ does not admit an analytical expression for arbitrary DMCs,[4] it can be obtained numerically via Arimoto's algorithm in [1]. Therefore, we can always numerically determine the upper and lower bounds for $E_J(Q,W)$.

Lemma 1 If we denote $\widehat{\rho} \triangleq \arg\max_{\rho\ge0}E(\rho)$, then the JSC sphere-packing bound of Proposition 1 is attained for rate $\widehat{R}_m = H(Q^{(\widehat{\rho})})$, where distribution $Q^{(\alpha)}$, $\alpha\ge0$, is defined by $Q^{(\alpha)}(s) \triangleq Q^{\frac{1}{1+\alpha}}(s)/(\sum_{s'\in\mathcal{S}}Q^{\frac{1}{1+\alpha}}(s'))$, $s\in\mathcal{S}$. Furthermore, if we let $\widetilde{\rho}\triangleq\min(\widehat{\rho},1)$, then the JSC random-coding bound of Proposition 1 is attained for rate $\widetilde{R}_m = H(Q^{(\widetilde{\rho})})$, $s\in\mathcal{S}$.

We know that if the lower (or upper) bound in Proposition 1 is attained for rate R' no less than R_{cr}, where R_{cr} is the channel critical rate, then $E_J(Q,W)$ is determined exactly [2]. In light of this fact, Theorem 1 and Lemma 1, we obtain the following explicit (computable) conditions.

Lemma 2 Define distribution Q^* by $Q^*(s)\triangleq Q^{(1)}(s)$, $s\in\mathcal{S}$. Then the following hold.

- $H(Q^*)\ge R_{cr} \Longleftrightarrow \widehat{\rho}\le1 \Longleftrightarrow \widehat{R}_m=\widetilde{R}_m\ge R_{cr}$. Thus, $E_J(Q,W)=E(\widehat{\rho})$.
- $H(Q^*)< R_{cr} \Longleftrightarrow \widehat{\rho}>1 \Longleftrightarrow \widehat{R}_m>\widetilde{R}_m=H(Q^*)$. Thus, $E(1)\le E_J(Q,W)\le E(\widehat{\rho})$.

We also have examined Csiszár's JSC expurgated lower bound using a similar approach, and we have partially addressed the computation of Csiszár's bounds for the (lossy) JSC exponent with distortion threshold [3]. Finally, in [5], we provide a systematic comparison of $E_J(Q,W)$ and the tandem exponent $E_T(Q,W)$, the exponent resulting from concatenating optimal source and channel codes. Sufficient conditions for which $E_J(Q,W) > E_T(Q,W)$ are also established.

References

[1] S. Arimoto, "Computation of random coding exponent functions," *IEEE Trans. Inform. Theory*, pp. 665-671, Nov. 1976.

[2] I. Csiszár, "Joint source-channel error exponent," *Probl. Contr. Inform. Theory*, vol. 9, pp. 315-328, 1980.

[3] I. Csiszár, "On the error exponent of source-channel transmission with a distortion threshold," *IEEE Trans. Inform. Theory*, vol. 28, pp. 823-828, Nov. 1982.

[4] R. G. Gallager, *Information Theory and Reliable Communication*, New York: Wiley, 1968.

[5] Y. Zhong, F. Alajaji, and L. L. Campbell, "When is joint source-channel coding worthwhile: An information theoretic perspective," *Proc. 22nd Bienn. Symp. Commun.*, Canada, June 2004.

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[2] We thus call the lower bound the "JSC random-coding bound" and the upper bound the "JSC sphere-packing bound."

[3] We assume that $H(Q) < C$, since otherwise $E_J(Q,W)=0$.

[4] Note that for *symmetric* DMCs (in the Gallager sense [4]), $E_o(\rho)$ can be analytically solved, hence yielding closed-form parametric expressions for the bounds in (1).