Bonferroni-Type Bounds on the Frame and Bit Error Rates of Coded AWGN and Block Rayleigh Fading Channels

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Abstract — **We present a method to find Bonferroni-type upper and lower bounds for the end-to-end frame error rate and bit error rate of a general communication system. The KAT lower bound of [2] is also studied. Knowledge of a particular structure in the system, such as the weight distribution of the channel code, can be used to speed-up the computations or reduce complexity. Tightness of the bounds is demonstrated for various channel codes and signaling schemes.**

I. Introduction

The KAT bound derived in [2] and the algorithmic Bonferroni bounds in [3, 10, 1] are used to find tight lower and upper bounds on the symbol error rate and bit error rate (BER) of uncoded data sent over PSK/QAM modulated AWGN, Rayleigh fading, and space-time orthogonal block coded channels, respectively. In most real-life applications, the data is channel coded and also packetized; hence another performance parameter of interest is the frame error rate (FER).

We consider a communication system with any type of channel coding and any complex signaling scheme. We establish good bounds on the system FER and BER for the case of additive white Gaussian noise and block Rayleigh fading channels. This work has two main contributions. First, we show that tighter bounds can be obtained by considering the second order pairwise error probabilities (PEPs) as compared with, for example [7], where only the PEPs are considered. Second, we provide a general approach for the efficient computation of the FER and BER, which can be tailored to specific communication systems to speed-up the analysis.

II. Problem Definition

Let us consider a communication system which encodes groups of data bits and maps them to a sequence of constellation points. The above mapping may include *any* kind of channel coding, such as linear block codes or trellis coded modulation. Each of such sequences will be called a "frame" and denoted by $S_i = [s_1^i, ..., s_T^i], i = 1, \cdots, M$, where $m = \log_2(M)$ is the number of the data bits and $T = m/(r_c p)$ is the frame length in symbols, where r_c is the rate of the channel code and 2^p is the constellation size to which the s_t^i belong. We will assume that s_t^i is selected from a unit-energy constellation and weighted later at the modulator. When s_t^i is transmitted and for a transmit power of $E_s = r_c p E_b$, where E_b is the power per an uncoded bit, the input-output relationship of the channel is represented by $R_t = \sqrt{E_s} H s_t^i + N_t$, where R_t and N_t are the channel output and additive noise, respectively. The additive noise is assumed to be complex Gaussian with i.i.d. real and imaginary parts, each having distribution $\mathcal{N}(0, N_0/2)$. We denote such a distribution by $\mathcal{CN}(0, N_0)$. We herein consider two cases: $H = 1 = const.$, which indicates the AWGN channel, and H being i.i.d. $\mathcal{CN}(0,1)$ among frames, which is the block Rayleigh fading model. In the latter case, we assume that the receiver can accurately estimate the channel fading coefficient H. We will also assume that all frames are equally likely and that the receiver performs maximum likelihood (ML) detection, i.e., it chooses frame $\hat{S}_i$ via $\hat{S} = \arg\min_S \|R - \sqrt{E_s} H S\|^2$, so that

$$\text{FER} = \sum_{u=1}^{M} P(\epsilon|S_u)P(S_u) = \frac{1}{M}\sum_{u=1}^{M} P_u\left(\bigcup_{i\neq u} \epsilon_{iu}\right), \quad (1)$$

where $P(\cdot|S_u) \triangleq P_u(\cdot)$ is the conditional probability of error given that S_u was sent, and ϵ_{iu} indicates the event that S_i has a smaller ML metric than S_u. The BER is given by

$$\text{BER} = \frac{1}{M}\sum_{u=1}^{M} P_b(u),$$

where

$$\begin{aligned} P_b(u) &= \frac{1}{m}\sum_{j=1}^{M} H(j,u) P(\hat{S} = S_j | S = S_u) \\ &= \frac{1}{m}\sum_{j=1}^{M} H(j,u)\left(1 - P_u\left(\bigcup_{i\neq j} \epsilon_{ij}\right)\right), \quad (2) \end{aligned}$$

where $H(j,u)$ is the Hamming distance between the data bits corresponding to S_j and S_u. Equations (1) and (2) express both the FER and BER in terms of the probability of a union, which can be bounded using the Bonferroni bounds as explained in [3]. The Bonferroni bounds on the probability of a union depend on the pairwise error probabilities and the probability of the intersection of two such events. In the following, we derive the formulas for these probabilities.

III. The AWGN Channel

Let $S_u = [s_1^u, \cdots, s_T^u]$ be the u^{th} possible sequence of signals for a frame, and let $d_t^{i,u} = s_t^i - s_t^u$. We first consider the AWGN channel. It can be shown that the pairwise error probability between S_u and S_i is given by

$$P_u(\epsilon_{iu}) = P(S_u \to S_i) = Q\left(\Delta_{i,u}\right), \quad (3)$$

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where $\Delta_{i,u}^2 = \gamma_s \sum_t |d_t^{i,u}|^2$ and $Q(\cdot)$ is the Gaussian tail function. $\gamma_s = E_s/N_0$ for BPSK signaling and $\gamma_s = E_s/2N_0$ for two dimensional signaling. The probability of the intersection of two pairwise error events can be shown to be equal to

$$\begin{aligned} P_u(\epsilon_{iu} \cap \epsilon_{ju}) &= P(S_u \to S_i, S_u \to S_j) \\ &= \Psi\left(\rho_{uij}, \Delta_{i,u}, \Delta_{j,u}\right), \end{aligned} \tag{4}$$

where $\Psi(\rho, x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_x^\infty \int_y^\infty e^{-\frac{\tau^2 - 2\rho\tau\lambda + \lambda^2}{2(1-\rho^2)}} d\tau d\lambda$ and

$$\rho_{uij} = \frac{\sum_{t=1}^T \langle d_t^{i,u}, d_t^{j,u} \rangle}{\left(\sum_{t=1}^T |d_t^{i,u}|^2\right)^{\frac{1}{2}} \left(\sum_{t=1}^T |d_t^{j,u}|^2\right)^{\frac{1}{2}}},$$

where $\langle x, y \rangle$ is the standard inner product. For BER bounds, we need $P_u(\epsilon_{ij}) = Q\left(\frac{\Delta_{i,u}^2 - \Delta_{j,u}^2}{\Delta_{i,j}}\right)$ and

$$P_u(\epsilon_{ij} \cap \epsilon_{kj}) = \Psi\left(\rho_{jik}, \frac{\Delta_{i,u}^2 - \Delta_{j,u}^2}{\Delta_{i,j}}, \frac{\Delta_{k,u}^2 - \Delta_{j,u}^2}{\Delta_{k,j}}\right).$$

IV. The Block Rayleigh Fading Channel

For the block Rayleigh fading channel, we use the results of [1] to evaluate the expected value of (3) and (4) with respect to H; that is, $P_u(\epsilon_{iu}) = E_H\left\{Q\left(|H|\Delta_{i,u}\right)\right\}$, which results in

$$P_u(\epsilon_{iu}) \stackrel{\triangle}{=} \Lambda(\Delta_{i,u}) = \frac{1}{2}\left(1 - \frac{\Delta_{i,u}}{\sqrt{2 + \Delta_{i,u}^2}}\right),$$

and

$$\begin{aligned} P_u\left(\epsilon_{iu} \cap \epsilon_{ju}\right) &= \frac{1}{2\pi}\int_0^{\varphi(\Delta_{i,u}/\Delta_{j,u},\rho)} \frac{2\sin^2\theta}{\Delta_{i,u}^2 + 2\sin^2\theta} d\theta \\ &+ \frac{1}{2\pi}\int_0^{\varphi(\Delta_{j,u}/\Delta_{i,u},\rho)} \frac{2\sin^2\theta}{\Delta_{j,u}^2 + 2\sin^2\theta} d\theta, \end{aligned}$$

where $\varphi(x, \rho) = \tan^{-1}\left(\frac{x\sqrt{1-\rho^2}}{1-\rho x}\right)$. We denote the above probability by $\Omega(\rho_{uij}, \Delta_{i,u}, \Delta_{j,u})$. For BER bounds, one can obtain $P_u(\epsilon_{ij}) = \Lambda\left((\Delta_{i,u}^2 - \Delta_{j,u}^2)/\Delta_{i,j}\right)$ and

$$P_u(\epsilon_{ij} \cap \epsilon_{kj}) = \Omega\left(\rho_{jik}, \frac{\Delta_{i,u}^2 - \Delta_{j,u}^2}{\Delta_{i,j}}, \frac{\Delta_{k,u}^2 - \Delta_{j,u}^2}{\Delta_{k,j}}\right).$$

V. Linear Block Codes and BPSK Signaling

Since the number of the PEPs grows exponentially in m, the computational complexity of the above bounds becomes prohibitive for large data blocks. In important special cases, such as when linear block codes and binary signaling (such as BPSK) are used, it is possible to consider only the all-zero codeword to reduce the complexity by a factor of M, which is very significant. Furthermore, the fact that all of the codewords with the same weight have the same PEP with the all-zero codeword can be used to reduce the computations considerably. In [8], the lower bound in [5] is used to derive a lower bound on the FER using only the weight distribution of the underlying code. The lower bound of [5] is also improved in [4] via the introduction of a weight function to be optimized. However, the evaluation of the algorithmic Bonferroni bounds may still be tedious for large m.

Our alternative is to use codewords up to a certain weight. Studying the Bonferroni bounds, one can note that using a subset of the codebook loosens these bounds. As will be shown in the next section, the performance loss will not be significant for the lower bound. The upper bound is loose only for codes operating on large data blocks.

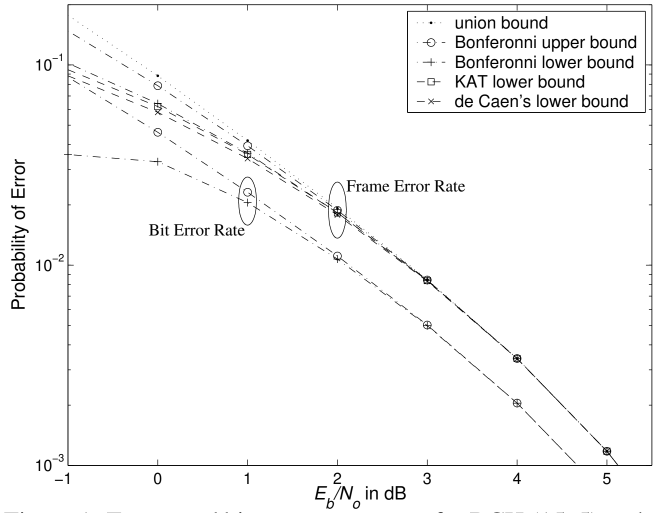

Figure 1: Frame and bit error rate curves for BCH (15, 5) code, 8-PSK modulation, and AWGN channel.

VI. Numerical Results

We consider a system with a uniform i.i.d. binary source, various channel codes, and 2- and 1-dimensional signaling. A typical plot for a two dimensional signaling scheme is shown in Figure 1, where the incoming bits are BCH (15, 5) coded, 8-PSK modulated, and sent over the AWGN channel. The entire codebook is used to calculate the bounds for this figure. We consider the Bonferroni upper and lower bounds, and the KAT [2] and de Caen [5] lower bounds and observe that the Bonferroni lower bound is the tightest among the lower bounds.

Figures 2 and 3 compare the tightness of the FER bounds for the case where only a subset of the codebook is used to compute the bounds for Golay (23, 12) and BCH (63, 24) codes, respectively. BPSK modulation is used, hence the lower bound of [8] can also be computed. For the Golay (BCH) code, only codewords with weight less than 7 (15) are considered in the computation of the KAT and Bonferroni-type bounds. Such a selection allows us to compute these bounds using 253 (651) codewords, instead of using 2047 (16777216) codewords, which is computationally not feasible. The Bonferroni and KAT lower bounds are still tight, but the Bonferroni upper bound is loose at low E_b/N_0. Poltyrev's upper bound [6] is superior to the Bonferroni upper bound used on the subset, although they are both the same as the union bound at FER values of interest (FER $< 10^{-3}$). Note first that the Bonferroni upper bound is easier to compute than Poltyrev's; therefore, it is the choice at FER values of interest. Second, the

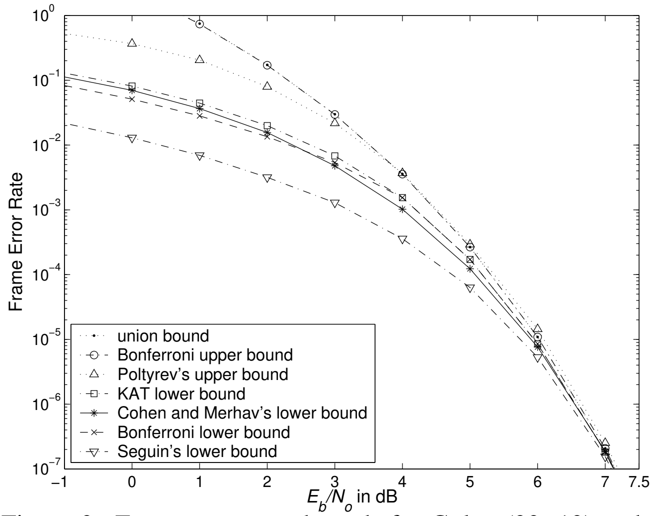

Figure 2: Frame error rate bounds for Golay (23, 12) code, BPSK signaling, and AWGN channel.

KAT and Bonferroni lower bounds are tighter than Seguin's bound; hence they are also tighter than Shannon's lower bound [9] at least at medium to high E_b/N_0. Also notice that the KAT bound is always tighter than the dot-product lower bound of [4].

Figures 4 and 5 demonstrate the performance of the Bonferroni, KAT, and Seguin bounds for the Rayleigh fading channel for Hamming (7, 4) and BCH (15, 5) codes. The lower bound is still tight, but the upper bound is not as tight for the Rayleigh fading channel, suggesting that the higher order probabilities are significant in the block fading case. As for the BER, we observe in our calculations that the Bonferroni upper bound is tight, but the lower bound is too loose to be useful.

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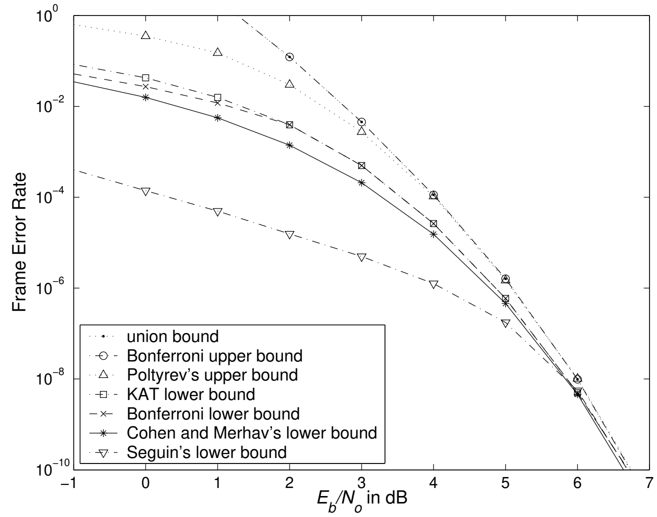

Figure 3: Frame error rate bounds for BCH (63, 24) code, BPSK signaling, and AWGN channel.

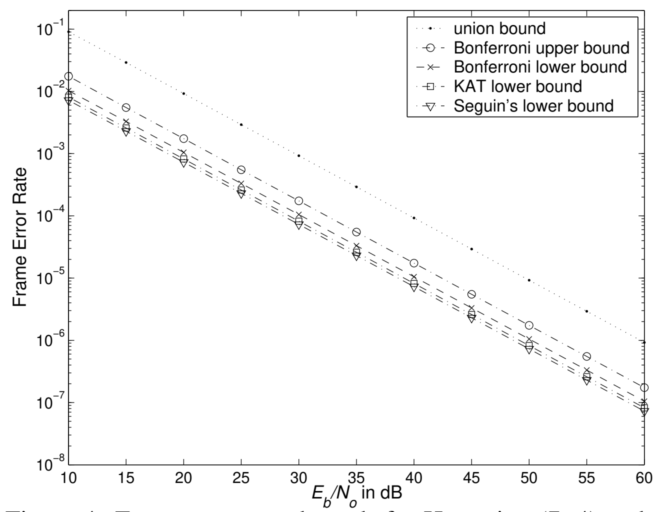

Figure 4: Frame error rate bounds for Hamming (7, 4) code, BPSK signaling, and Rayleigh fading channel.

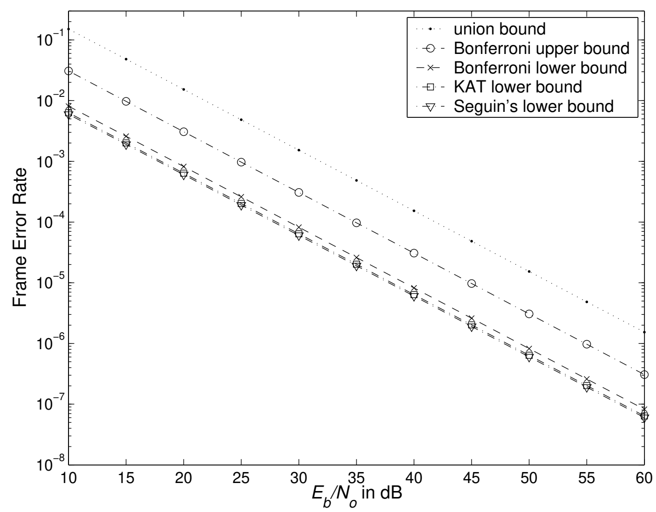

Figure 5: Frame error rate bounds for BCH (15, 5) code, BPSK signaling, and Rayleigh fading channel.