On Bounding the Union Probability

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***Abstract*—We present new results on bounding the probability of a finite union of events, $P\left(\bigcup_{i=1}^N A_i\right)$ for a fixed positive integer N, using partial information on the events joint probabilities. We first consider bounds that are established in terms of $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$ where $c_1, \ldots, c_N$ are given weights. We derive a new class of lower bounds of at most pseudo-polynomial computational complexity. This class of lower bounds generalizes the recent bounds in [1], [2] and can be tighter in some cases than the Gallot-Kounias [3]–[5] and Prékopa-Gao [6] bounds which require more information on the events probabilities. We next consider bounds that fully exploit knowledge of $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. We establish new numerical lower/upper bounds on the union probability by solving a linear programming problem with $\frac{(N-1)^3+N+3}{2}$ variables. These bounds coincide with the optimal lower/upper bounds when $N \leq 7$ and are guaranteed to be sharper than the optimal lower/upper bounds of [1], [2] that use $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$.**

***Index Terms*–Union probability, upper and lower bounds, linear programming, probability of error analysis, communication systems.**

I. Introduction

Lower/upper bounds on the union probability $P\left(\bigcup_{i=1}^N A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s were actively investigated in the recent past. The optimal bounds can be obtained numerically by solving linear programming (LP) problems with 2^N variables [6], [7]. Since the number of variables is exponential in the number of events, N, some suboptimal but numerically efficient bounds were proposed, such as the bounds in [8] that employ the dual basic feasible solutions to reduce the complexity of the LP problem, and the algorithmic Bonferroni-type lower/upper bounds in [9], [10].

Among the established analytical bounds is the Kuai-Alajaji-Takahara lower bound (for convenience, hereafter referred to as the KAT bound) [11] that was shown to be better than the Dawson-Sankoff (DS) [12] and the D. de Caen (DC) [13] bounds. Noting that the KAT bound is expressed in terms of $\{P(A_i)\}$ and only the *sums* of the pairwise event probabilities, *i.e.*, $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, in order to fully exploit all pairwise event probabilities, it is observed in [14]–[16] that the analytical bounds can be further improved algorithmically by optimizing over subsets. Furthermore, in [6], the KAT bound is extended by using additional partial information such as the sums of joint probabilities of three events, i.e., $\{\sum_{j,l} P(A_i \cap A_j \cap A_l), i = 1, \ldots, N\}$. Recently, using the same partial information as the KAT bound, i.e., $\{P(A_i)\}$ and $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, the optimal lower/upper bound as well as a new analytical bound which is sharper than the KAT bound were developed in [1], [2].

In this paper, we first establish a new class of lower bounds on $P\left(\bigcup_{i=1}^N A_i\right)$ using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$ for a given weight or parameter vector $\boldsymbol{c} = (c_1, \ldots, c_N)^T$. These lower bounds are shown to have at most pseudo-polynomial computational complexity and to be sharper in certain cases than the existing Gallot-Kounias (GK) [3]–[5] and Prékopa-Gao (PG) [6] bounds, although the later bounds employ more information on the events joint probabilities. Furthermore, for bounds on $P\left(\bigcup_{i=1}^N A_i\right)$ that fully exploit knowledge of $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$, a new numerical lower/upper bound is proposed by solving an LP problem with $\frac{(N-1)^3+N+3}{2}$ variables. This numerical lower/upper bound is proven to be an optimal lower/upper bound when $N \leq 7$ and to be always better than the optimal lower/upper bound which uses $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. Finally, we should note that these general union probability bounds can be applied to effectively estimate and analyze the error performance of a variety of coded or uncoded communication systems (e.g., see [2], [9], [10], [14], [17]–[22]).

II. New Bounds using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$

For simplicity, and without loss of generality, we assume the events $\{A_1, \ldots, A_N\}$ are in a finite probability space $(\Omega, \mathscr{F}, P)$, where N is a fixed positive integer. Let $\mathscr{B}$ denote the collection of all non-empty subsets of $\{1, 2, \ldots, N\}$. Given $B \in \mathscr{B}$, we let ω_B denote the atom in the union $\cup_{i=1}^N A_i$ such that for all $i = 1, \cdots, N$, $\omega_B \in A_i$ if $i \in B$ and $\omega_B \notin A_i$ if $i \notin B$ (note that some of these "atoms" may be the empty set). For ease of notation, for a singleton $\omega \in \Omega$, we denote $P(\{\omega\})$ by $p(\omega)$ and $p(\omega_B)$ by p_B. Since $\{\omega_B : i \in B\}$ is the collection of all the atoms in A_i, we have $P(A_i) = \sum_{\omega \in A_i} p(\omega) = \sum_{B \in \mathscr{B}: i \in B} p_B$, and

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$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B\in\mathscr{B}} p_B. \quad (1)$$

Suppose there are N functions $f_i(B), i = 1, \ldots, N$ such that $\sum_{i=1}^{N} f_i(B) = 1$ for any $B \in \mathscr{B}$ (i.e., for any atom ω_B). If we further assume that $f_i(B) = 0$ if $i \notin B$ (i.e., $\omega_B \notin A_i$), we can write

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B\in\mathscr{B}} \left(\sum_{i=1}^{N} f_i(B)\right) p_B = \sum_{i=1}^{N} \sum_{B\in\mathscr{B}:i\in B} f_i(B) p_B. \quad (2)$$

Note that if we define

$$f_i(B) = \begin{cases} \frac{1}{|B|} = \frac{1}{\deg(\omega_B)} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \quad (3)$$

where the degree of ω, $\deg(\omega)$, is the number of A_i's that contain ω, then $\sum_{i=1}^{N} f_i(B) = 1$ is satisfied and (2) becomes

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \sum_{\omega\in A_i} \frac{p(\omega)}{\deg(\omega)}. \quad (4)$$

Note that many of the existing bounds, such as the DC bound [13] and KAT bound [11] and the bounds in [1] [2], are based on (4).

In the following lemma, we propose a generalized expression of (4). To the best of our knowledge this lemma is novel.

Lemma 1: Suppose $\{\omega_B, B \in \mathscr{B}\}$ are all the $2^N - 1$ atoms in $\bigcup_i A_i$. If $\boldsymbol{c} = (c_1, \ldots, c_N)^T \in \mathbb{R}^N$ satisfies

$$\sum_{k\in B} c_k \neq 0, \quad \text{for all} \quad B \in \mathscr{B} \quad (5)$$

then we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{N} A_i\right) &= \sum_{i=1}^{N} \sum_{B\in\mathscr{B}:i\in B} \frac{c_i p_B}{\sum_{k\in B} c_k} \\ &= \sum_{i=1}^{N} \sum_{\omega\in A_i} \frac{c_i p(\omega)}{\sum_{\{k:\omega\in A_k\}} c_k}. \end{aligned} \quad (6)$$

Proof: If we define

$$f_i(B) = \begin{cases} \frac{c_i}{\sum_{k\in B} c_k} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \quad (7)$$

where the parameter vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)^T$ satisfies $\sum_{k\in B} c_k \neq 0$ for all $B \in \mathscr{B}$ (therefore $c_i \neq 0, i = 1, \ldots, N$), then $\sum_i f_i(\omega) = 1$ holds and we can get (6) from (2). ■

Note that (6) holds for any $\boldsymbol{c}$ that satisfies (5) and is clearly a generalized expression of (4).

A. Relation to the Cohen-Merhav bound [19]

Let $m_i(\omega_B)$ be non-negative functions. Then by the Cauchy-Schwarz inequality,

$$\left[\sum_{B:i\in B} f_i(B) p_B\right]\left[\sum_{B:i\in B} \frac{p_B}{f_i(B)} m_i^2(\omega_B)\right] \geq \left[\sum_{B:i\in B} p_B m_i(\omega_B)\right]^2. \quad (8)$$

Thus, using (2), we have

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \frac{\left[\sum_{B:i\in B} p_B m_i(\omega_B)\right]^2}{\sum_{B:i\in B} \frac{p_B}{f_i(B)} m_i^2(\omega_B)}. \quad (9)$$

If we define $f_i(B)$ by (3), then (9) reduces to

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_i \frac{\left[\sum_{\omega\in A_i} p(\omega) m_i(\omega)\right]^2}{\sum_j \sum_{\omega\in A_i\cap A_j} p(\omega) m_i^2(\omega)}, \quad (10)$$

which is the Cohen-Merhav lower bound in [19, Theorem 2.1]; note that equality in (10) holds when $m_i(\omega) = \frac{1}{\deg(\omega)}$ (i.e., $m_i(\omega_B) = \frac{1}{|B|}$).

B. Relation to the GK Bound [3], [4]

In this subsection, we assume that the elements of $\boldsymbol{c}$ are positive, i.e., $\boldsymbol{c} \in \mathbb{R}_+^N$, and connect the GK bound [3] [4] with (6). The GK bound was recently revisited in [5] where it is reformulated as

$$\ell_{\text{GK}} = \max_{\boldsymbol{c}\in\mathbb{R}^N} \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i \sum_k c_i c_k P(A_i \cap A_k)}, \quad (11)$$

and the optimal $\boldsymbol{c}$ for (11), denoted by $\tilde{\boldsymbol{c}}$, can be computed by

$$\tilde{\boldsymbol{c}} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}, \quad (12)$$

where $\boldsymbol{\alpha} = (P(A_1), \ldots, P(A_N))^T$ and $\boldsymbol{\Sigma}$ is the $N\times N$ matrix whose (i, j)-th element is $P(A_i \cap A_j)$.

First, consider $\boldsymbol{c} \in \mathbb{R}_+^N$ fixed. Then, by the Cauchy-Schwarz inequality, we have

$$\left[\sum_{B:i\in B} \frac{c_i p_B}{\sum_{k\in B} c_k}\right]\left[\sum_{B:i\in B} \left(\frac{\sum_{k\in B} c_k}{c_i}\right) p_B\right] \geq P(A_i)^2. \quad (13)$$

Note that

$$\begin{aligned} \sum_{B:i\in B} \left(\frac{\sum_{k\in B} c_k}{c_i}\right) p_B &= \frac{1}{c_i} \sum_{k=1}^{N} \sum_{B:i\in B,k\in B} c_k p_B \\ &= \frac{\sum_k c_k P(A_i \cap A_k)}{c_i}. \end{aligned} \quad (14)$$

Then for all i,

$$\sum_{B:i\in B} \frac{c_i p_B}{\sum_{k\in B} c_k} \geq \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)} \quad (15)$$

By summing (15) over i, we get another new lower bound:

$$P\left(\bigcup_i A_i\right) \geq \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)}. \quad (16)$$

Note that we can use Cauchy-Schwarz Inequality again:

$$\left[\sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)}\right]\left[\sum_i c_i \sum_k c_k P(A_i \cap A_k)\right] \geq \left[\sum_i c_i P(A_i)\right]^2. \quad (17)$$

Since the above inequality holds for any positive $\boldsymbol{c}$, we have

$$P\left(\bigcup_i A_i\right) \geq \max_{\boldsymbol{c}\in\mathbb{R}_+^N} \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)} \geq \max_{\boldsymbol{c}\in\mathbb{R}_+^N} \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i \sum_k c_i c_k P(A_i \cap A_k)}. \quad (18)$$

Note that the lower bounds in (18) are weaker than the GK bound (11), however, if the optimal $\boldsymbol{c}$ of (11), $\tilde{\boldsymbol{c}}$, happen to satisfy $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$, then the bounds in (18) coincide with the GK bound (11).

C. New Class of Lower Bounds

We only consider $\boldsymbol{c} \in \mathbb{R}_+^N$ in this subsection. A new class of lower bounds is given in the following theorem.

Theorem 1: Defining $\mathscr{B}^- = \mathscr{B} \setminus \{1,\ldots,N\}$, $\tilde{\gamma}_i := \sum_k c_k P(A_i \cap A_k)$, $\tilde{\alpha}_i := P(A_i)$ and

$$\tilde{\delta} := \max_i \left[\frac{\tilde{\gamma}_i - \left(\sum_k c_k - \min_k c_k\right)\tilde{\alpha}_i}{\min_k c_k}\right]^+, \quad (19)$$

where $\boldsymbol{c} \in \mathbb{R}_+^N$, a class of lower bounds is given by

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \tilde{\delta} + \sum_{i=1}^N \ell_i'(\boldsymbol{c}, \tilde{\delta}), \quad (20)$$

where

$$\ell_i'(\boldsymbol{c},x) = [P(A_i) - x]\left(\frac{c_i}{\sum_{k\in B_1^{(i)}} c_k} + \frac{c_i}{\sum_{k\in B_2^{(i)}} c_k} - \frac{c_i \sum_k c_k \left[P(A_i\cap A_k) - x\right]}{[P(A_i)-x]\left(\sum_{k\in B_1^{(i)}} c_k\right)\left(\sum_{k\in B_2^{(i)}} c_k\right)}\right), \quad (21)$$

and

$$B_1^{(i)} = \arg\max_{\{B\in\mathscr{B}^-: i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad \text{s.t.} \quad \frac{\sum_{k\in B} c_k}{c_i} \leq \frac{\sum_k c_k \left[P(A_i\cap A_k)-x\right]}{c_i\left[P(A_i)-x\right]},$$
$$B_2^{(i)} = \arg\min_{\{B\in\mathscr{B}^-: i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad \text{s.t.} \quad \frac{\sum_{k\in B} c_k}{c_i} \geq \frac{\sum_k c_k \left[P(A_i\cap A_k)-x\right]}{c_i\left[P(A_i)-x\right]}. \quad (22)$$

Proof: Let $x = p_{\{1,2,\ldots,N\}}$ and consider $\sum_i \ell_i'(\boldsymbol{c},x) + x$ as a new lower bound where where $\ell_i'(\boldsymbol{c},x)$ equals to the objective value of the problem

$$\min_{\{p_B: i\in B, B\in\mathscr{B}^-\}} \sum_{B: i\in B, B\in\mathscr{B}^-} \frac{c_i p_B}{\sum_{k\in B} c_k}$$
$$\text{s.t.} \quad \sum_{B: i\in B, B\in\mathscr{B}^-} p_B = P(A_i) - x,$$
$$\sum_{B: i\in B, B\in\mathscr{B}^-} \left(\frac{\sum_{k\in B} c_k}{c_i}\right) p_B = \frac{1}{c_i}\sum_k c_k\left[P(A_i\cap A_k) - x\right],$$
$$p_B \geq 0, \quad \text{for all} \quad B\in\mathscr{B}^- \quad \text{such that} \quad i\in B. \quad (23)$$

The solution of (23) exists if and only if

$$\min_k c_k \leq \frac{\tilde{\gamma}_i - \left(\sum_k c_k\right)x}{\tilde{\alpha}_i - x} \leq \sum_k c_k - \min_k c_k. \quad (24)$$

Therefore, the new lower bound can be written as

$$\min_x \left[x + \sum_{i=1}^N \ell_i'(\boldsymbol{c},x)\right] \quad \text{s.t.}$$
$$\left[\frac{\tilde{\gamma}_i - \left(\sum_k c_k - \min_k c_k\right)\tilde{\alpha}_i}{\min_k c_k}\right]^+ \leq x \leq \frac{\tilde{\gamma}_i - (\min_k c_k)\tilde{\alpha}_i}{\sum_k c_k - \min_k c_k}, \forall i. \quad (25)$$

We can prove that the objective function of (25) is non-decreasing with x. Therefore, defining $\tilde{\delta}$ as in (19), the new lower bound can be written as (20) where $\ell_i'(\boldsymbol{c},\tilde{\delta})$ can be obtained by solving (23), which is given in (21). ■

Remark 1: Note that the problems in (22) are exactly the $0/1$ knapsack problem with mass equals to value [23], which can be computed in pseudo-polynomial time, and can be arbitrarily closely approximated by an algorithm running in polynomial time [23].

Remark 2: It can readily be shown that if $\boldsymbol{c} = \kappa\mathbf{1}$ for any non-zero constant κ with $\mathbf{1}$ being the all-one vector of length N, the new lower bound reduces to the analytical lower bound in [1], [2], which is sharper than the KAT bound. It can also be shown that if the optimal $\tilde{\boldsymbol{c}}$ of the GK bound satisfies $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$, then the new lower bound is sharper than the GK bound.

III. New Bounds using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$

In this section, we derive new numerical lower/upper bounds for $P\left(\bigcup_{i=1}^N A_i\right)$ using $\{P(A_i)\}$ and $\{P(A_i\cap A_j)\}$. First, consider the p_B's in (1) as variables. Then the following (exhaustive) LP problem with 2^N variables gives the optimal lower/upper bound established using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$:

$$\min_{\{p_B, B\in\mathscr{B}\}} \Big/ \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B$$
$$\text{s.t.} \quad \sum_{i,j\in B, B\in\mathscr{B}} p_B = P(A_i\cap A_j), \quad i,j\in\{1,\ldots,N\},$$
$$p_B \geq 0, B\in\mathscr{B}. \quad (26)$$

The optimality of (26) can be easily proved by showing its achievability: for each p_B, construct an atom ω_B such that $p(\omega_B) = p_B$ and let $\omega_B \in A_i, \forall i \in B$. However, the computational complexity of the optimal lower/upper bound in (26) is exponential. Next, we consider a relaxed problem of (26), which is given in the following:

$$
\begin{aligned}
&\min_{\{p_B, B\in\mathscr{B}\}} \Big/ \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B,\\
\text{s.t.}\quad & \sum_{i,j\in B, B\in\mathscr{B}} p_B = P(A_i\cap A_j), \quad i,j\in\{1,\dots,N\},\\
& \sum_{B:i,j,l\in B,|B|=k} p_B \ge 0, \qquad \sum_{B:i,j\in B,l\notin B,|B|=k} p_B\ge 0,\\
& \sum_{B:i\in B,j,l\notin B,|B|=k} p_B \ge 0, \qquad \sum_{B:i,j,l\notin B,|B|=k} p_B\ge 0,\\
& \forall i,j,l,k\in\{1,\dots,N\}.
\end{aligned} \tag{27}
$$

Since the solution of (27) is a lower/upper bound for the union probability $P\left(\bigcup_{i=1}^N A_i\right)$, we next show that the solution of (27) can be obtained by solving an LP problem with $\frac{(N-1)^3+N+3}{2}$ variables, which coincides with the optimal lower/upper bounds when $N\le 7$. The main results are in the following.

Lemma 2: The solution of problem (27) coincides with the optimal lower/upper bound in (26) when $N\le 7$.

Lemma 3: The problem (27) shares the same solution with the following LP:

$$
\begin{aligned}
&\min_{\{p_B, B\in\mathscr{B}\}} \Big/ \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B,\\
\text{s.t.}\quad & \sum_{i,j\in B, B\in\mathscr{B}} p_B = P(A_i\cap A_j), \quad i,j\in\{1,\dots,N\},\\
& \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j\in B,l\notin B,|B|=k} p_B\ge 0,\\
& \sum_{B:l\in B,i,j\notin B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B\ge 0,\\
& \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B\ge 0,\\
& \sum_{B:i,j\in B,l\notin B,|B|=k} p_B + \sum_{B:l\in B,i,j\notin B,|B|=k} p_B\ge 0,\\
& \sum_{B:i,j\in B,|B|=k} p_B + \sum_{B:i\in B,j,l\notin B,|B|=k} p_B\ge 0,\\
& \forall i,j,l,k\in\{1,\dots,N\}.
\end{aligned} \tag{28}
$$

Theorem 2: Defining $a_{ij}(k)=\sum_{i,j\in B,|B|=k} p_B$, the LP problem (28) can be reformulated as an LP of $\{a_{ij}(k)\}$ (i.e., N^3 variables). The number of variables can hence be reduced from N^3 to $\frac{(N-1)^3+N+3}{2}$.

Proof: Define $a(k)=\sum_{|B|=k} p_B$ and $a_i(k)=\sum_{i\in B,|B|=k} p_B$, then it can be readily shown that $a(k)=\sum_{i=1}^N \frac{a_i(k)}{k}$ and $a_i(k)=\sum_{j=1}^N \frac{a_{ij}(k)}{k}$. Therefore, both $a(k)$ and $a_i(k)$ are linear functions of $\{a_{ij}(k)\}$.

We next demonstrate that the number of variables can be reduced from N^3 to $\frac{(N-1)^3+N+3}{2}$. Note that according to the definition of $a_{ij}(k)$, we have: i) $a_{ij}(1)=P(\{x\in A_i\cap A_j, \deg(x)=1\})=0, \forall i\neq j$; ii) $a_{ij}(k)=a_{ji}(k)$; iii) $a_{ij}(N)=P\left(\bigcap_{i=1}^N A_i\right)$ for any i and j. Therefore, the number of variables for different values of k can be reduced to

$$
\begin{cases} N & \text{if } k=1\\ \frac{N(N-1)}{2} & \text{if } k=2,\dots,N-1\\ 1 & \text{if } k=N \end{cases} \tag{29}
$$

Thus, the total number of variables is $N+\frac{N(N-1)(N-2)}{2}+1$.

Now it is suffices to show that the objective function and all the constraints in (28) can be written as functions of $a_{ij}(k)$ so that all $\{p_B\}$ can be replaced using $a_{ij}(k)$. In the following, we directly give the results, which one can easily verify.

The objective function and the first constraint of (28) can be written as

$$
\begin{aligned}
&\sum_k\sum_i\sum_j \frac{a_{ij}(k)}{k^2} = \sum_{B\in\mathscr{B}} p_B,\\
&\sum_k a_{ij}(k) = \sum_{i,j\in B,B\in\mathscr{B}} p_B = P(A_i\cap A_j), \quad \forall i,j.
\end{aligned} \tag{30}
$$

Finally, for all $i,j,l,k\in\{1,\dots,N\}$, the other constraints of (28) as functions of $\{p_B\}$ can be written as functions of $\{a_{ij}(k)\}$ as follows:

$$
\begin{aligned}
&a_{ij}(k) = \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j\in B,l\notin B,|B|=k} p_B,\\
&a(k)-a_i(k)-a_j(k)+a_{ij}(k)\\
&\quad = \sum_{B:l\in B,i,j\notin B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B,\\
&a(k)-a_l(k)-a_i(k)-a_j(k)+a_{ij}(k)+a_{il}(k)+a_{jl}(k)\\
&\quad = \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B,\\
&a_l(k)+a_{ij}(k)-a_{il}(k)-a_{jl}(k)\\
&\quad = \sum_{B:i,j\in B,l\notin B,|B|=k} p_B + \sum_{B:l\in B,i,j\notin B,|B|=k} p_B,\\
&a_i(k)-a_{ij}(k) = \sum_{B:i,l\in B,j\notin B,|B|=k} p_B + \sum_{B:i\in B,j,l\notin B,|B|=k} p_B.
\end{aligned} \tag{31}
$$

Therefore, the lower/upper bounds of (27) can be solved by an LP with $\frac{(N-1)^3+N+3}{2}$ variables. ■

Remark 3: According to Lemma 2, the new numerical lower/upper bound coincides with the optimal lower/upper bounds in (26) when $N\le 7$. Furthermore, we can show that the new numerical lower/upper bounds are sharper than the numerical bounds in [1], [2], which have been proved to be the optimal lower/upper bounds in terms of $\{P(A_i)\}$ and $\{\sum_j P(A_i\cap A_j)\}$.

IV. Numerical Examples

Due to the space limitation, we only present lower bounds in this section. The same eight systems as in [1] are used and the corresponding results are shown in Table I. For comparison, we include bounds that utilize $\{P(A_i)\}$ and

TABLE I

COMPARISON OF LOWER BOUNDS (* INDICATES $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$ AND A BOLD NUMBER INDICATES COINCIDENCE WITH THE OPTIMAL BOUND (26)).

System	I	II*	III*	IV	V	VI	VII	VIII*
N	6	6	6	7	3	4	4	4
$P\left(\bigcup_{i=1}^N A_i\right)$	0.7890	0.6740	0.7890	0.9687	0.3900	0.3252	0.5346	0.5854
KAT Bound [11]	0.7247	0.6227	0.7222	0.8909	0.3833	0.2769	0.4434	0.5412
GK Bound [3], [4]	0.7601	0.6510	0.7508	0.9231	0.3813	0.2972	0.4750	0.5390
PG Bound [6]	0.7443	0.6434	0.7556	0.9148	**0.3900**	0.3240	*0.5281*	*0.5726*
Analytical Bound [2, Eq. (7)]	0.7247	0.6227	0.7222	0.8909	**0.3900**	0.3205	0.4562	0.5464
Numerical Bound [2, Eq. (5)]	0.7487	0.6398	0.7427	0.9044	**0.3900**	**0.3252**	**0.5090**	0.5531
New Bound (20) with $\boldsymbol{c} = \tilde{\boldsymbol{c}}^+$	0.7638	0.6517	0.7512	0.9231	**0.3900**	0.2951	0.4905	0.5412
New Bound (20) with random $\boldsymbol{c}$	0.7783	0.6633	0.7810	0.9501	**0.3900**	0.3203	0.4992	0.5666
Stepwise Bound [9]	**0.7890**	**0.6740**	**0.7890**	**0.9687**	**0.3900**	0.3027	0.5009	**0.5673**
New Numerical Bound (27)	**0.7890**	**0.6740**	**0.7890**	**0.9687**	**0.3900**	**0.3252**	**0.5090**	**0.5673**

$\{\sum_j P(A_i \cap A_j), i = 1, \ldots, N\}$, such as the KAT bound [11], the analytical bound in [1], [2], and the numerical optimal bound in this class [1], [2]. We also include the GK bound [3], [4] and the stepwise bound [9], which fully exploit $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. The PG lower bound [6], which extends the KAT bound by using $\{P(A_i)\}$, $\{\sum_j P(A_i \cap A_j)\}$ and $\{\sum_{j,l} P(A_i \cap A_j \cap A_l)\}$, is also investigated in the examples. The Cohen-Merhav bound (10) [19] is not included since it is not clear how to choose the function $m_i(\omega)$ in our examples.

For the proposed bound (20) we consider two cases for choosing $\boldsymbol{c}$. The first choice for $\boldsymbol{c}$, denoted by $\tilde{\boldsymbol{c}}^+$, has components $\tilde{c}_i^+ = \max(\tilde{c}_i, \epsilon)$ with $\tilde{\boldsymbol{c}}$ given in (12) and $\epsilon > 0$ close to zero. Therefore, if $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$ then $\tilde{\boldsymbol{c}}^+ = \tilde{\boldsymbol{c}}$, so that in this case the new bound (20) is guaranteed to be sharper than the GK bound. If $\tilde{\boldsymbol{c}} \notin \mathbb{R}_+^N$, on the other hand, we still have $\tilde{\boldsymbol{c}}^+ \in \mathbb{R}_+^N$. The second choice of $\boldsymbol{c}$ is to randomly generate $\boldsymbol{c} \in \mathbb{R}_+^N$ and compute (20). In the examples, we generate 1000 values for $\boldsymbol{c}$ and show the largest obtained value for (20).

From Table I, one remarks that for Systems II, III and VIII we have $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$, so that the new bound (20) with $\boldsymbol{c} = \tilde{\boldsymbol{c}}$ is sharper than the GK bound, as expected. Also, the new bound (20) can be further improved by randomly generating additional $\boldsymbol{c}$ values as shown in the table. Furthermore, the PG bound which uses sums of joint probabilities of three events, may be even poorer (e.g., see Systems I and VI) than the numerical bound in [1], [2] which utilizes less information but is optimal in the class of lower bounds using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. It is also weaker than (20) in several cases (see Systems I-IV). Finally, our numerical bound (27) is always sharper than the other tested bounds, and coincides with the optimal bound (26) with exponential complexity in N since $N < 7$ holds for these examples.

REFERENCES

[1] J. Yang, F. Alajaji, and G. Takahara, "Lower bounds on the probability of a finite union of events," submitted, 2014. [Online]. Available: http://arxiv.org/abs/1401.5543

[2] ——, "New bounds on the probability of a finite union of events," in *2014 IEEE International Symposium on Information Theory (ISIT)*, June 2014, pp. 1271–1275.

[3] S. Gallot, "A bound for the maximum of a number of random variables," *Journal of Applied Probability*, vol. 3, no. 2, pp. 556–558, 1966.

[4] E. G. Kounias, "Bounds for the probability of a union, with applications," *The Annals of Mathematical Statistics*, vol. 39, no. 6, pp. 2154–2158, 1968.

[5] C. Feng, L. Li, and J. Shen, "Some inequalities in functional analysis, combinatorics, and probability theory," *The Electronic Journal of Combinatorics*, vol. 17, no. R58, p. 1, 2010.

[6] A. Prékopa and L. Gao, "Bounding the probability of the union of events by aggregation and disaggregation in linear programs," *Discrete Applied Mathematics*, vol. 145, no. 3, pp. 444–454, 2005.

[7] P. Veneziani, "Lower bounds of degree 2 for the probability of the union of N events via linear programming," June 2007, unpublished.

[8] B. Vizvári, "New upper bounds on the probability of events based on graph structures," *RUTCOR Research Report (Sept. 2004)*.

[9] H. Kuai, F. Alajaji, and G. Takahara, "Tight error bounds for nonuniform signaling over AWGN channels," *IEEE Transactions on Information Theory*, vol. 46, no. 7, pp. 2712–2718, 2000.

[10] F. Behnamfar, F. Alajaji, and T. Linder, "Tight error bounds for space-time orthogonal block codes under slow Rayleigh flat fading," *IEEE Transactions on Communications*, vol. 53, no. 6, pp. 952–956, 2005.

[11] H. Kuai, F. Alajaji, and G. Takahara, "A lower bound on the probability of a finite union of events," *Discrete Mathematics*, vol. 215, no. 1-3, pp. 147–158, 2000.

[12] D. Dawson and D. Sankoff, "An inequality for probabilities," *Proceedings of the American Mathematical Society*, vol. 18, no. 3, pp. 504–507, 1967.

[13] D. De Caen, "A lower bound on the probability of a union," *Discrete Mathematics*, vol. 169, no. 1, pp. 217–220, 1997.

[14] F. Behnamfar, F. Alajaji, and T. Linder, "An efficient algorithmic lower bound for the error rate of linear block codes," *IEEE Transactions on Communications*, vol. 55, no. 6, pp. 1093–1098, 2007.

[15] F. M. Hoppe, "Improving probability bounds by optimization over subsets," *Discrete Mathematics*, vol. 306, no. 5, pp. 526–530, 2006.

[16] ——, "The effect of redundancy on probability bounds," *Discrete Mathematics*, vol. 309, no. 1, pp. 123–127, 2009.

[17] G. Seguin, "A lower bound on the error probability for signals in white Gaussian noise," *IEEE Transactions on Information Theory*, vol. 44, no. 7, pp. 3168–3175, Nov. 1998.

[18] S. Yousefi and A. K. Khandani, "A new upper bound on the ML decoding error probability of linear binary block codes in AWGN interference," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3026–3036, Dec. 2004.

[19] A. Cohen and N. Merhav, "Lower bounds on the error probability of block codes based on improvements on de caen's inequality," *IEEE Transactions on Information Theory*, vol. 50, no. 2, pp. 290–310, 2004.

[20] H. Nguyen and N. Tran, "Bonferroni-type bounds for CDMA systems with nonuniform signalling," *IEEE Commun. Lett.*, vol. 9, no. 7, pp. 583–585, July 2005.

[21] R. Bettancourt, L. Szczecinski, and R. Feick, "BER evaluation of BICM-ID via Bonferroni-type bounds," *IEEE Trans. Veh. Technol.*, vol. 57, no. 5, pp. 2815–2821, Sep. 2008.

[22] Z. Mao, J. Cheng, and J. Shen, "A new lower bound on error probability for nonuniform signals over AWGN channels," in *Wireless Communications and Networking Conference (WCNC)*. IEEE, 2013, pp. 3005–3009.

[23] V. V. Vazirani, *Approximation Algorithms*. New York, NY, USA: Springer-Verlag New York, Inc., 2001.