A Queue-Based Model for Wireless Rayleigh Fading Channels with Memory

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***Abstract*— We investigate the modeling of a family of hard-decision frequency-shift keying demodulated correlated Rayleigh fading channels using a recently introduced queue-based channel (QBC) model for binary communication channels with memory. The QBC parameters are estimated by minimizing the Kullback-Leibler divergence rate between the distributions of error sequences generated by the QBC and the fading channels and the modeling accuracy is evaluated in terms of autocorrelation function and channel capacity. Numerical results indicate that the QBC provides a very good approximation of the fading channels for a wide range of channel conditions.**

I. Introduction

It is well known that most real-world communication channels have memory and experience noise and fading distortions in a bursty fashion [6]. Thus memory is exhibited by the channels in terms of statistical dependence in their error process. In order to design effective communication systems for such channels, it is of paramount importance to fully understand their behavior. This is accomplished via channel modeling, where the primary objective is to provide a model whose properties are both complex enough to closely capture the real channel statistical characteristics, and simple enough to allow mathematically tractable system analysis.

During the past several decades, a variety of channel models have been proposed and studied for the modeling of wireless channels (e.g., [9]). One of the earliest models for channels with memory is the Gilbert-Elliott channel (GEC) [4], [2], which belongs to the family of finite-state Markov channels (FSMC's) [3, pp. 97-111]. A common feature of FSMC's is that they are constructed based on a finite-state hidden Markov chain [6]. However, due to their HMM structure, it is difficult to obtain single-letter analytical expressions for their statistical and information theoretical quantities (such as capacity and/or block transition probability) in terms of the channel parameters. Thus, they often do not allow mathematically tractable analysis, particularly when incorporated within an overall source and/or channel coded system. Indeed, to date, few coding techniques have been successfully constructed for such HMM based channel models and for channels with memory in general [7]. It is therefore vital to construct channel models which can represent well the behavior of real-world channels while remaining analytically tractable for design purposes.

The queue-based channel (QBC), recently introduced in [11], is a binary additive noise channel with memory based on a finite queue. It features a stationary ergodic Mth order Markov noise source and it is fully characterized by four parameters (ϵ, α, p and M). The channel admits single-letter expressions for its block transition distribution and capacity, which is an attractive feature for mathematical analysis and code design. It is also important to point out that Pimentel, Falk and Lisbôa recently showed in a numerical study [10] that the class of binary channel models with additive Kth order Markov noise (to which the QBC belongs) is a good approximation, in terms of the autocorrelation function (ACF) and variational distance, to the family of hard-decision frequency-shift keying demodulated correlated Rayleigh fading channels for a good range of fading environments, particularly for medium and fast fading rates. Note however, that the Kth order Markov noise channels considered in [10] have a complexity (number of parameters) that grows exponentially with K, rendering it impractical for the modeling of channels with large memory such as very slow Rayleigh fading channels (e.g., see Fig. 3 or [10, Fig. 11]). The QBC model, on the other hand, does not suffer from this limitation as it is fully described by only four parameters and it can accommodate very large values of the memory M. In a recent related work [12], the problem of modeling the GEC using the QBC was investigated, and it was shown (numerically) that the QBC provides a good approximation of the GEC for various channel conditions. We herein study the problem of approximating the same class of Rayleigh fading channels studied in [10] via the QBC.

The rest of this paper is organized as follows. Preliminaries on the GEC and QBC channel models are presented in Section II. In Section III, we investigate the modeling of hard-decision demodulated correlated Rayleigh fading channels [10] via the QBC. In Section IV, we provide the numerical fitting results. For the sake of comparison, we also model the fading channels via the GEC (which has the same number of parameters as the QBC) using the parameterization method of Pimentel *et. al.* in [10]. The accuracy of both methods is

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evaluated in terms of ACF and capacity. In Section V, we conclude with a summary along with a direction for future work.

II. THE GEC AND QBC BINARY CHANNEL MODELS

Hereafter, a discrete-time binary additive noise communication channel refers to a channel with common input, noise and output alphabet $\mathcal{X} = \mathcal{Z} = \mathcal{Y} = \{0, 1\}$ described by $Y_n = X_n \oplus Z_n$, for $n = 1, 2, 3, \cdots$, where $\oplus$ denotes addition modulo 2, and where X_n, Z_n, and Y_n denote, respectively, the channel's input, noise, and output at time n. Hence a transmission error occurs at time n whenever $Z_n = 1$. It is assumed that the input and noise sequences are independent of each other. In this work, a given noise process $\{Z_n\}_{n=1}^{\infty}$ will be generated according to one of the GEC, the QBC and the "discretized" Rayleigh fading channel.

A. Gilbert-Elliott Channel

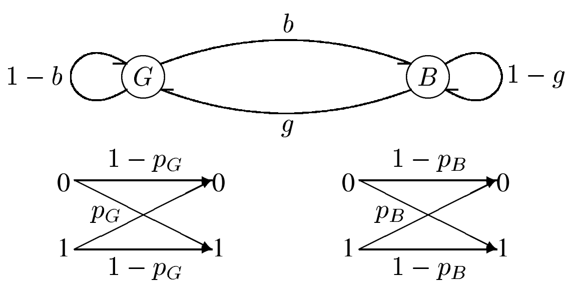

Fig. 1. The Gilbert-Elliott channel model

The GEC model is driven by an underlying stationary ergodic Markov chain with two states: a good state and a bad state, denoted by G (or state 0) and B (or state 1), see Fig. 1. In a fixed state, the channel behaves like a binary symmetric channel (BSC). The GEC is thus a time-varying BSC, where p_G and p_B are the crossover probabilities in the good and bad states, respectively (the Gilbert channel (GC) [4] is obtained when $p_G = 0$; i.e., it behaves like a noiseless BSC in the good state). After every channel transmission, the chain makes a state transition according to the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 1-b & b \\ g & 1-g \end{bmatrix},$$

where $0 < b < 1$ and $0 < g < 1$. A useful approach for calculating the probability of an error or noise sequence for the GEC is discussed in [9]. The probability of a noise sequence of length n, $z^n = (z_1, z_2, \cdots, z_n)$, can be expressed as

$$\mathrm{P}_{\text{GEC}}(z^n) = \boldsymbol{\pi}^T \left(\prod_{k=1}^{n} \boldsymbol{P}(z_k) \right) \mathbf{1}, \tag{1}$$

where $\mathbf{P}(0)$ and $\mathbf{P}(1)$ are each a 2×2 matrix whose (i, j)th entry is the probability that the output symbol is 0 and 1, respectively, when the chain makes a transition from state $s_{k-1} = i$ to $s_k = j$. The vector $\boldsymbol{\pi} = [\pi_0 \quad \pi_1]^T$ indicates the stationary distribution vector of the underlying Markov chain, and $\mathbf{1}$ is the 2-dimensional vector with all ones.

B. Queue-Based Channel with Memory

The additive noise process of the queue-based binary channel with memory [11] is generated according to a sampling mechanism involving the following two parcels.

- **Parcel 1** is a queue of length M as shown in Fig. 2, that contains initially M balls, either red or black.

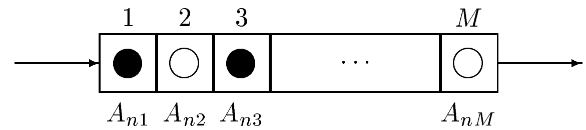

Fig. 2. A queue of length M.

The random variables A_{nk} (n is a time index referring to the nth experiment, $n \geq 1$; k represents the position in the queue, $k = 1, 2, \cdots, M$) are defined by:

$$A_{nk} = \begin{cases} 1, & \text{if the } k\text{th cell contains a red ball,} \\ 0, & \text{if the } k\text{th cell contains a black ball.} \end{cases}$$

- **Parcel 2** is an urn that contains a very large number of balls where the proportion of black balls is $1 - p$ and the proportion of red balls is p, where $p \in (0, 1)$, $p \ll 1/2$.

We assume that the probability of selecting parcel 1 (the queue) is ε, while the probability of selecting parcel 2 (the urn) is $1 - \varepsilon$ and $\varepsilon \in [0, 1)$. Notice that the channel is actually a BSC with crossover probability p when $\varepsilon = 0$, in which case we experiment on the urn only.

The noise process $\{Z_n\}_{n=1}^{\infty}$ is generated according to the following procedure. By flipping a biased coin (with Pr(Head)=ε), we select one of the two parcels (select the queue if Heads and the urn if Tails). If parcel 2 (the urn) is selected, a pointer randomly points at a ball, and identifies its color. If parcel 1 (the queue) is selected, the procedure is determined by the length of the queue. If $M \geq 2$, a pointer points at the ball in cell k with probability $1/(M - 1 + \alpha)$, for $k = 1, 2, \cdots, M - 1$ and $\alpha \geq 0$, and points at the ball in cell M with probability $\alpha/(M - 1 + \alpha)$, and identifies its color. If $M = 1$, a pointer points at the ball in the only cell of the queue with probability 1; i.e., $\alpha = 1$. If the selected ball from either parcel is red (respectively black), we introduce a red (respectively black) ball in cell 1 of the queue, pushing the last ball in cell M out. The noise process $\{Z_n\}_{n=1}^{\infty}$ is then modeled as follows:

$$Z_n = \begin{cases} 1, & \text{if the } n\text{th experiment points at a red ball,} \\ 0, & \text{if the } n\text{th experiment points at a black ball.} \end{cases}$$

By studying the channel state process $\{\underline{S}_n\}_{n=1}^{\infty}$, where $\underline{S}_n \stackrel{\triangle}{=} (A_{n1}, A_{n2}, \cdots, A_{nM})$, it can be shown that the channel noise process $\{Z_n\}_{n=1}^{\infty}$ is a stationary ergodic (irreducible) Mth order Markov process. As a result, various statistical and information theoretic quantities of the QBC, such as the channel block transition probability, capacity and ACF, can be determined (in closed-form) in terms of M, p, ε, and α (see [11] for the detailed expressions). It should be also noted that the finite-memory Polya contagion channel introduced in [1] is a special case of the QBC (obtained by setting $\alpha = 1$).

III. Fitting Rayleigh Fading Channels

We consider a discrete (binary-input, binary-output) communication system, referred to as the discrete channel with Clarke's autocorrelation (DCCA) model, that employs binary frequency-shift keying modulation, a time-correlated Rayleigh flat-fading channel, and a hard quantized noncoherent demodulation [10]. The combination of digital modulator, fading channel, and digital demodulator yields the equivalent DCCA model. The study and analysis of the statistical behavior of the DCCA model is important since the design and construction of effective error control schemes for this simplified (binary-input, binary-output) model helps us better exploit the system memory and achieve reliable communication over the underlying correlated fading channel.

The QBC [11] is next used to model the equivalent binary error sequence of the DCCA, which represents the successes and failures that result from the transmission of symbols over the above fading channel.

A. QBC Parameter Estimation

For a given DCCA, we construct a QBC whose noise or error process is statistically "close" in the Kullback-Leibler sense to the noise process generated by the DCCA. Specifically, given a DCCA with fixed average signal-to-noise ratio (SNR) E_s/N_0, and normalized Doppler frequency $f_D T$ resulting in bit error rate BER_{DCCA} and correlation coefficient Cor_{DCCA}, we estimate the QBC parameters M, p, ε, and α that minimize the Kullback-Leibler divergence rate (KLDR) $\lim_{n\to\infty}(1/n)D_n(\text{P}_{\text{DCCA}} \parallel \text{P}_{\text{QBC}}^{(M)})$, subject to the constraints $\text{BER}_{\text{QBC}} = \text{BER}_{\text{DCCA}}$ and $\text{Cor}_{\text{QBC}} = \text{Cor}_{\text{DCCA}}$, where $(1/n)D_n(\text{P}_{\text{DCCA}} \parallel \text{P}_{\text{QBC}}^{(M)})$ is the normalized Kullback-Leibler divergence (NKLD) between the n-fold DCCA and QBC noise distributions, P_{DCCA} and $\text{P}_{\text{QBC}}^{(M)}$, respectively:

$$D_n(\text{P}_{\text{DCCA}} \parallel \text{P}_{\text{QBC}}^{(M)}) = \sum_{z^n \in \{0,1\}^n} \text{P}_{\text{DCCA}}(z^n) \log_2 \frac{\text{P}_{\text{DCCA}}(z^n)}{\text{P}_{\text{QBC}}^{(M)}(z^n)}.$$

The expression for $\text{P}_{\text{QBC}}^{(M)}$ is given in closed form in [11], while the expression for P_{DCCA} can be directly obtained from [10, Eq. (3) with $\Omega = 1$].

It can be shown (e.g., see [5]) that the KLDR between the DCCA and QBC noise processes does exist and can be expressed as

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} D_n(\text{P}_{\text{DCCA}} \parallel \text{P}_{\text{QBC}}^{(M)}) \\ &= -\mathcal{H}_{\text{DCCA}}(Z) - E_{\text{P}_{\text{DCCA}}}[\log_2 \text{P}_{\text{QBC}}^{(M)}(Z_{M+1}|Z^M)], \qquad (2)\end{aligned}$$

where $\text{P}_{\text{QBC}}^{(M)}(z_{M+1}|z^M)$ is the QBC conditional error probability of symbol $M+1$ given the previous M symbols and $\mathcal{H}(\cdot)$ denotes the entropy rate,

$$\mathcal{H}(Z) = \lim_{n\to\infty} \frac{1}{n} H(Z_1, Z_2, \cdots Z_n).$$

Then the minimization reduces to maximizing the second term in (2) (which is independent of n) over the QBC parameters. Note that in our approximation, we match BER and Cor of both channels to guarantee identical noise marginal distributions and identical probabilities of two consecutive errors (ones). Hence, given these constraints, the above optimization problem reduces to an optimization over only two QBC parameters.

B. GEC Parameter Estimation

For a given DCCA, Pimentel *et. al.* provided a method to estimate the GEC parameters in [10]. It is shown that the GEC parameters can be calculated solving the linear and nonlinear systems of the probability of a finite set of error sequences generated by the DCCA with length no longer then 3 (see [10]).

IV. Modeling Results and Discussions

We evaluate how well the QBC model fits or approximates the DCCA according to two criteria: channel capacity and ACF. The QBC ACF and capacity expressions are provided in [11]. The ACF of the DCCA can be obtained directly from [10, Eq. (3)]:

$$R_{\text{DCCA}}[m] = \frac{1}{\left(2 + \frac{E_s}{N_0}\right)^2 - \left(\frac{E_s}{N_0}\rho(m)\right)^2}$$

where

$$\rho(m) = J_0(2\pi m f_D T),$$

and $J_0(x) = \sum_{k=0}^{\infty}(-1)^k(\frac{x^k}{2^k k!})^2$ is the zero-order Bessel function of the first kind.

It can be shown that the capacity of the DCCA is given by

$$C_{\text{DCCA}} = 1 - \mathcal{H}_{\text{DCCA}}(Z).$$

The entropy rate of the DCCA error process is not known in closed form. However, we can approximate it by computing $(1/n)H(Z^n)$ for large values of n and thus obtain a lower bound on C_{DCCA}, given by

$$C_{\text{DCCA},n} \stackrel{\triangle}{=} 1 - \frac{1}{n} H_{\text{DCCA}}(Z^n).$$

In our calculations, we use values of n as large as 21.

For the sake of comparison, we also present modeling results via the GEC using the method of Pimentel *et. al.* in [10]. As noted in Section I, in [10], the authors also employ arbitrary Kth order Markov noise models to approximate the fading channels. However, unlike our QBC model which has only four parameters (as the GEC) and allows large values for its memory order M (since its noise is a specially structured Mth order Markov process generated by our queue scheme), the Kth order Markov models of [10] are unstructured and hence suffer from the limitation of having a number of parameters that grows exponentially[1] with K. Therefore, with the exception of the comparison with the Markov models of

[1] As a result, only models with memory order up to 6 are studied in [10]. Such models are shown to approximate well channels with fast and medium fading rates ($f_D T > 0.02$); but they are inadequate for slow fading rates. As we later show in this section, the QBC model can accommodate large values of the memory order; thus, it can provide a good approximation of channels with slow fading ($f_D T < 0.02$) in addition to medium and fast fading.

[10] made in Fig. 3, we herein mainly compare our QBC-based modeling method with the GEC-based modeling method of [10] since both channels have identical number of parameters, hence identical degrees of freedom and complexity.

The GEC capacity is obtained via the algorithm in [7]. The ACF of the GEC can also be obtained directly from (1):

$$R_{\text{GEC}}[m] = \boldsymbol{\pi}^T \boldsymbol{P}(1) \left(\prod_{k=1}^{m-1} \boldsymbol{P} \right) \boldsymbol{P}(1)\mathbf{1}, \tag{3}$$

where $\boldsymbol{\pi}$, $\boldsymbol{P}(1)$, and $\boldsymbol{P}$ are defined in Section II-A.

A wide range of DCCA channel parameters is investigated with SNR = 15 dB and 25 dB, $f_D T = 0.001$, 0.005, 0.01 and 0.1. The SNR and $f_D T$ values (except for $f_D T = 0.005$) were chosen to match the conditions of the correlated Rayleigh fading channels studied in [10].

Typical modeling results in terms of the ACF for the DCCA, its QBC approximation and its GEC approximation are shown in Figs. 3-7. We observe a strong ACF agreement between the QBC and the DCCA in these figures (this behavior was indeed observed for all computations, especially for $f_D T = 0.1$ (Fig. 7) where the ACF curve of the DCCA and its QBC approximation are identical). For very slow and slow fading (Figs. 3-5), the ACF curve for the GEC takes a longer span of m before eventually converging, which indicates that the GEC (as fitted in [10]) might not be adequate for modeling slow Rayleigh fading ($f_D T = 0.001$ and 0.005). We also observe that the QBC has better performance than the Markov models in [10] (see Fig. 3), but with significantly smaller complexity since it is fully described by four parameters.

Note that since the QBC noise is a homogeneous Markov process, the KLDR between the DCCA and the QBC error processes exists and admits a simple expression. Hence, it is practical to minimize this KLDR by maximizing the expected value in (2) over the QBC parameters which is independent of n (see Section III-A). However, this approach is not easily applicable to the GEC since the KLDR between the DCCA and the GEC noise processes does not admit a simple expression in general (as the GEC noise is hidden Markovian). The method of parameterization of the GEC in [10] is simple, but it only takes into account error sequences no longer than 3, which implies that this method is not appropriate for approximating slow fading.

Our results show that the largest Markovian memory M for the QBC model that best fits the DCCA is 20, while the largest Markovian memory K for the general Markov noise channel model considered in [10] is 6 (higher order Markov models could not be obtained in [10] due to their prohibitive exponential complexity). This explains why the QBC is more suitable for fitting slow fading with large memory than the Markov noise model considered in [10].

Modeling results in terms of capacity are shown in Fig. 8, where the lower bound for the capacity of the DCCA and the capacities of the QBC approximation and the GEC approximation are shown for different SNR values and $f_D T$ values. We clearly observe from the figures that the capacity curves of the QBC and the lower bound curves for the capacity of the DCCA match quite well, and the capacities for $f_D T = 0.1$ are almost identical. The last observation can be explained by the fact that the DCCA has low memory at $f_D T = 0.1$ (fast fading); hence the lower bound for its capacity is tight (since $\frac{1}{n} H(Z^n) = H(Z_1)$ if Z^n is memoryless). Overall, we observe a strong match in capacity between the DCCA and its QBC approximation. In terms of capacity, the GEC has nearly as good a performance as the QBC in fitting the DCCA.

V. Summary

In this work, we approximate hard-decision demodulated correlated Rayleigh fading channels (represented by the DCCA) via the QBC model. Numerical results show a strong agreement between the ACF and capacity curves of the QBC and the DCCA. This leads us to conclude that the QBC provides a very good approximation of the DCCA under a variety of channel conditions. The QBC provides a much better performance in terms of fitting the DCCA than the GEC and the Markov models of [10] for the range of slow and very slow fading. An important feature of this queue-based channel model is that it is valuable for characterizing a wide class of communication channels with memory, while remaining mathematically simple and flexible.

One possible direction for future work is the design, construction and analysis of powerful channel codes for the QBC. One important objective in this problem is the judicious design of the channel codes in order to fully exploit the channel memory. Some results in this direction involving low density parity check (LDPC) codes are reported in [8].

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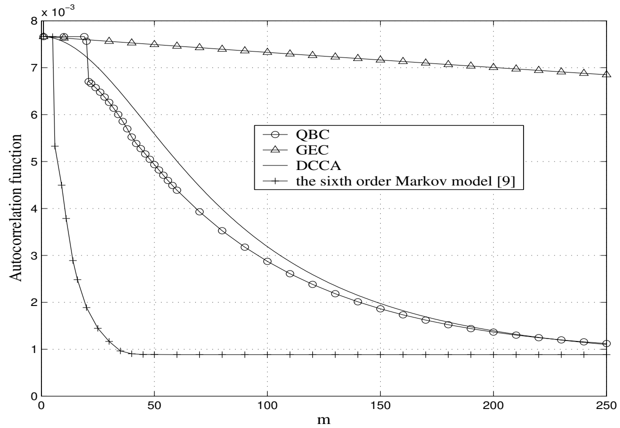

Fig. 3. DCCA fitting via the QBC: Autocorrelation function vs m for Rayleigh fading channel: SNR = 15 dB and normalized Doppler frequency $f_D T = 0.001$.

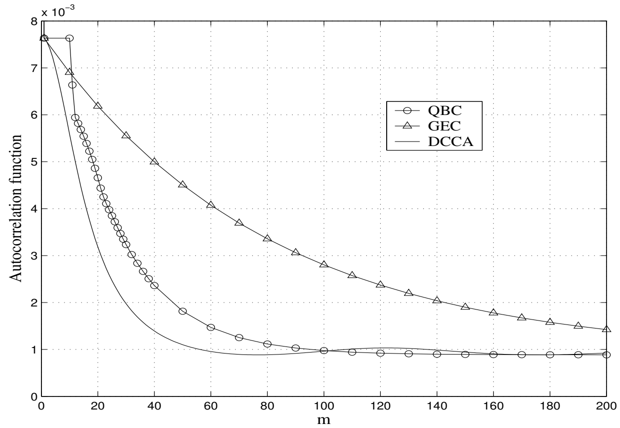

Fig. 4. DCCA fitting via the QBC: Autocorrelation function vs m for Rayleigh fading channel: SNR = 15 dB and normalized Doppler frequency $f_D T = 0.005$.

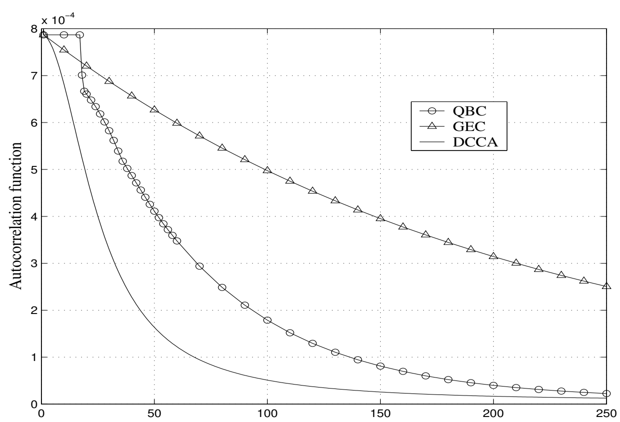

Fig. 5. DCCA fitting via the QBC: Autocorrelation function vs m for Rayleigh fading channel: SNR = 25 dB and normalized Doppler frequency $f_D T = 0.001$.

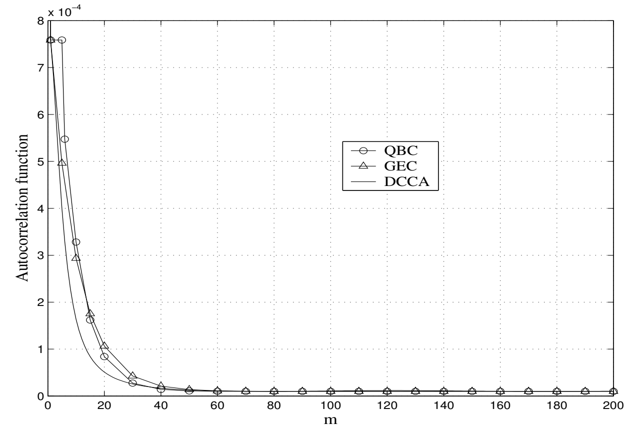

Fig. 6. DCCA fitting via the QBC: Autocorrelation function vs m for Rayleigh fading channel: SNR = 25 dB and normalized Doppler frequency $f_D T = 0.005$.

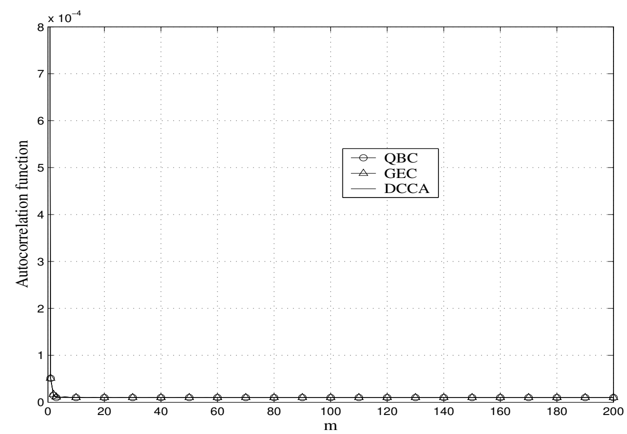

Fig. 7. DCCA fitting via the QBC: Autocorrelation function vs m for Rayleigh fading channel: SNR = 25 dB and normalized Doppler frequency $f_D T = 0.1$.

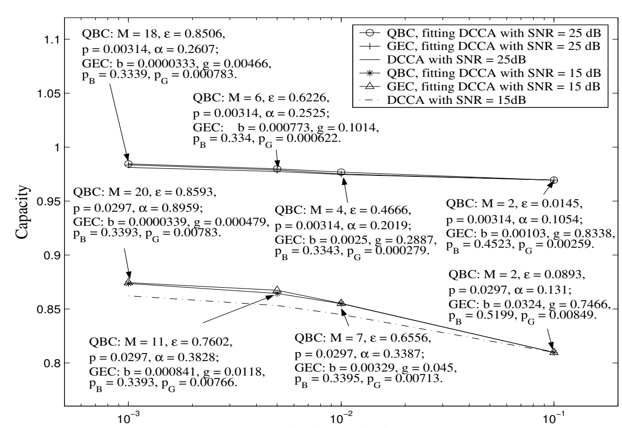

Fig. 8. DCCA fitting via the QBC: Capacity vs normalized Doppler frequency $f_D T$ for Rayleigh fading channel.