EXERCISES ON TORIC VARIETIES

Graduate Summer School for program in Combinatorial Algebraic Geometry

(1) Let $T = (k^*)^n$. Show that $\mathrm{Hom}(T, k^*) \cong \mathbb{Z}^n$.

(2) Show that a monoid S is finitely generated if and only if the associated semigroup algebra $k[S]$ is finitely generated.

(3) (a) Let σ be a pointed rational polyhedral cone in $\mathbb{R}^2$. How do you geometrically construct the dual cone $\sigma^\vee$? Find $\sigma_i^\vee$ when $\sigma_1 = \langle(1,0),(0,1)\rangle$, $\sigma_2 = \langle(1,0),(1,1)\rangle, \sigma_3 = \langle(1,0),(-1,1)\rangle$.
   (b) Show that if σ is a polyhedral convex cone, then $\sigma^\vee$ is a polyhedral convex cone.
   (c) Let σ be a polyhedral convex cone. Then $\sigma^{\vee\vee} = \sigma$.
   (d) If σ is a a rational convex polyhedral cone, then $\sigma^\vee$ is rational.

(4) (a) Let σ be a pointed rational polyedral cone and let $\tau \leq \sigma$ be a face of σ. Then τ is a pointed rational polyhedral cone.
   (b) Let σ be a rational convex polyhedral cone and let $\tau \leq \sigma$ be a face. Then there is $u \in \mathrm{relint}(\tau^\perp \cap \sigma^\vee) \cap M$ such that $\tau = \sigma \cap u^\perp$.

(5) If $\sigma = \langle e_1, \ldots, e_n\rangle$, then $U_\sigma = \mathbb{A}^n$.

(6) Let $\sigma = \langle v_1, \ldots, v_r\rangle$ be a rational polyhedral convex cone. Prove Gordan's lemma, i.e., that S_σ is a finitely generated monoid. Hint: Show that there are finitely many lattice points contained in the set $\{a_1v_1 + \cdots + a_rv_r \mid 0 \leq a_i \leq 1\}$ and show that these generate.

(7) Let S_σ be generated by $u_1, \ldots, u_r$, so $k[S_\sigma] = k[\chi^{u_1}, \ldots, \chi^{u_t}] = k[Y_1, \ldots, Y_t]/I$; show that I is generated by binomials of the from $Y_1^{a_1} \cdots Y_t^{a_t} - Y_1^{b_1} \cdots Y_t^{b_t}$, where $a_1, \ldots, a_t, b_1, \ldots, b_t$ are nonnegative integers such that $a_1u_1 + \cdots + a_tu_t = b_1u_1 + \cdots b_tu_t$.

(8) Let $\sigma = \langle e_1, v\rangle$. For which v is σ a smooth pointed rational polyhedral cone? Classify all smooth affine toric surfaces.

(9) Show that the glueing in the construction of a toric variety form a fan is compatible.

(10) For the following fans Δ in $\mathbb{R}^2$, find $X(\Delta)$. (Here we only give the maximal cones of Δ).
   (a) $\Delta = \{\langle e_1, e_2\rangle, \langle e_2, -e_1\rangle\}$.
   (b) $\Delta = \{\langle e_1, e_2\rangle, \langle e_2, -e_1\rangle, \langle -e_1, -e_2\rangle, \langle -e_2, e_1\rangle\}$.
   (c) $\Delta = \{\langle e_1, e_2\rangle, \langle e_2, -e_1 - e_2\rangle, \langle -e_1 - e_2, e_1\rangle\}$.
   (d) $\Delta = \{\langle e_1, e_1 + e_2\rangle, \langle e_1 + e_2, e_2\rangle\}$.
   (e) $\Delta = \{\langle e_1, e_1 + e_2\rangle, \langle e_1 + e_2, e_2\rangle, \langle e_2, -e_1 - e_2\rangle, \langle -e_1 - e_2, e_1\rangle\}$.
   (f) $\Delta = \{\langle e_1, e_2\rangle, \langle e_2, -e_1 + ae_2\rangle, \langle -e_1 + ae_2, -e_2\rangle, \langle -e_2, e_1\rangle\}$. (This is the Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(a))$. If you haven't met this surface before, simply take this as the definition).

(11) For the following polytopes P determine Δ_P and $X(\Delta_P)$.
   (a) $P = \mathrm{conv}\langle(0,0),(1,0)\rangle$.
   (b) $P = \mathrm{conv}\langle(0,0),(1,0),(0,1),(1,1)\rangle$.
   (c) $P = \mathrm{conv}\langle(0,0),(1,0),(0,1)\rangle$.

(12) If τ spans $N_{\mathbb{R}}$, show that x_τ is the unique fixed point of the action of the torus on T_N on U_τ. If τ does not span $N_{\mathbb{R}}$, show that there is no torus fixed point in U_τ. Show that $\dim O_\tau = \mathrm{codim}_\tau N_{\mathbb{R}}$.

(13) Show that $U_\sigma = \sqcup_{\tau \leq \sigma} O_\tau$. Hint: For a point $x \in U_\sigma$, show that $x^{-1}(k^*) = \sigma^\vee \cap \tau^\perp \cap M$ for some face τ of σ. Then $x \in O_\tau$.

(14) Let Δ be fan. Assume that Δ is complete, i.e., $|\Delta| = \cup\sigma = \mathbb{R}^n$. Show that $V(\tau) = \mathbb{P}^1$ for every cone τ of codimension 1 in Δ.

(15) Let $X = \mathrm{Bl}_p\mathbb{P}^2$. Find ψ_D for $D = E, H, E + H$.

(16) Show that $P_{mD} = mP_D$ for all positive integers m. If $u \in M$, then $P_{D+\mathrm{div}(\chi^u)} = P_D - u$. Moreover, if D, E are Weil-divisors, then $P_D + P_E \subset P_{D+E}$.

(17) Let $X = \mathbb{P}^2$. Find $\mathrm{Pic}(X)$.

(18) Let $X = \mathrm{Bl}_p\mathbb{P}^2$. Find $\mathrm{Pic}(X)$.

(19) Let $X = \mathbb{P}^1$ and let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1$ with $\overline{\Psi}_{\mathcal{L}} = \{u_0, u_1\}$. Then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(u_1 - u_0)$.

(20) Let $P = \mathrm{conv}\langle 0, 3\rangle$.
   (a) Find the defining equations of $X(\Delta_P)$ in $\mathbb{P}^3$.
   (b) Show that the ideal I of this variety can be generated by the 2×2 minors of a 2×3 matrix. (What is the matrix?)
   (c) Find the minimal free resolution of I.