TORIC VARIETIES

I. Introduction to toric varieties

<u>Def</u> Let k be a field. A <u>toric variety</u> is an equivariant partial compactification of the algebraic torus $T = (k^*)^d$.

Note that to T is associated its character lattice

$$M = \mathrm{Hom}(T, k^*) \underset{\substack{\uparrow \\ \text{exercise}}}{\cong} \mathbb{Z}^d$$

The strength of toric varieties is the existence of a dictionary between algebra geometric properties of X and combinatorial properties of polytopes or fans that are contained in M or its dual lattice, $N = \mathrm{Hom}(M, \mathbb{Z})$.

Often it is easier to compute on the combinatorial side and the goal of this course is to give you some tools to do that.

<u>References</u>: Cox-Little-Schenck

Fulton

Mustaţă - online

Affine toric varieties

Def A monoid is a set S with an operation $+$ that is commutative, associative, and has a unit element 0.

Examples • $(\mathbb{N}^d,+)$, $(\mathbb{Z}^d,+)$, $(k,\cdot)$ are monoids.

Def A morphism of monoids is a map $\varphi: S \to S'$ between two monoids s.t. $\varphi(u_1+u_2) = \varphi(u_1)+\varphi(u_2)$ for all $u_1,u_2 \in S$ and $\varphi(0)=0$.

To a monoid $S \subseteq \mathbb{Z}^d$ is associated an algebra $k[S]$. It has a basis over k given by the characters associated to $u \in S$. For $u=(u_1,\dots,u_d)$, $\chi^u = x_1^{u_1}\cdots x_d^{u_d}$ & $k[S] = \bigoplus_{u\in S} k\chi^u$.
Multiplication is defined by $\chi^{u_1}\cdot\chi^{u_2} = \chi^{u_1+u_2}$, and we let $1=\chi^0$.
If $\varphi: S\to S'$ is a morphism of monoids, then there is an induced morphism of k-algebras $f: k[S] \to k[S']$ defined by $f(\chi^u) = \chi^{\varphi(u)}$.

Exercise S is finitely generated if and only if $k[S]$ is finitely generated.

Example: • If $S=(\mathbb{N},+)$, then $k[S] \cong k[t]$. Indeed, for $m\in\mathbb{N}$, we let $f(\chi^m)=t^m$! This extends to an isomorphism.

• If $S=(\mathbb{N}^d,+)$, then $k[S] = k[t_1,\dots,t_d]$ polynomial ring $(e_i \leftrightarrow t_i)$

• If $S=(\mathbb{Z}^d,+)$, then $k[S] = k[t_1^{\pm 1},\dots,t_d^{\pm 1}]$ ring of Laurent polynomials

• If $S=(\mathbb{N}^r\times\mathbb{Z}^{d-r},+)$, then $k[S] = k[t_1,\dots,t_r, t_{r+1}^{\pm1},\dots,t_S^{\pm1}]$

Some convex geometry

Let V be a vector space over $\mathbb{R}$. A subset $C \subset V$ is called <u>convex</u> if for every $v_1, v_2 \in C$ and $0 \leq \lambda \leq 1$, we have $\lambda v_1 + (1-\lambda) v_2 \in C$. C is called a <u>cone</u> if for $v \in C$ and $\lambda \geq 0$ we have $\lambda v \in C$. Note that C is a <u>convex cone</u> if and only if it is a cone and for $v_1, v_2 \in C$, we have $v_1 + v_2 \in C$.

A <u>convex polytope</u> is the convex hull of finitely many points in V.

A <u>polyhedral convex cone</u> is a convex cone generated by finitely many vectors, i.e.,

$$C = \{ \lambda_1 v_1 + \dots + \lambda_r v_r \mid \lambda_i \geq 0 \} \quad \text{for some vectors } v_1, \dots, v_r \text{ in } V.$$

We write $C = \langle v_1, \dots, v_r \rangle$.

We say that a convex cone C is <u>pointed</u> if and only if whenever $v \in C$ and $-v \in C$, $v = 0$.

<u>Fact</u> Convex polytopes and polyhedral cones are closed in V.

Let $K \subset V$ be a closed convex subset. A <u>supporting hyperplane</u> is an affine hyperplane H s.t. $H \cap K \neq \emptyset$ and K is contained in at least one of the half-spaces determined by H. A <u>face</u> of K is the intersection of K with some supporting hyperplane.

A <u>facet</u> is a face of codim 1.

If σ is a polyhedral cone and τ a face of σ, we write $\tau \leq \sigma$.

Let U be the dual vector space to V. Let σ be a convex cone in V. The <u>dual cone</u> is defined to be

$$\sigma^\vee = \{ u \in U \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}$$

<u>Exercise</u> Show that if σ is a polyhedral convex cone, then $\sigma^\vee$ is a polyhedral convex cone and $(\sigma^\vee)^\vee = \sigma$.

Exercise Let τ, σ be polyhedral cones. Then $\tau \leq \sigma \Rightarrow \sigma^\vee \subseteq \tau^\vee$.

We say that $\sigma \subseteq N_{\mathbb{R}}$ is a rational polyhedral cone if $\sigma = \langle v_1, \dots, v_r \rangle$ with $v_i \in N$.

Exercise Let σ be a pointed rational polyhedral cone and let $\tau \leq \sigma$. Then τ is a pointed rational polyhedral cone.

Exercise If σ is a rational polyhedral cone, then $\sigma^\vee$ is rational.

Let σ be a cone in $N_{\mathbb{R}}$. The relative interior of σ is the topological interior of σ in the space $\mathbb{R}\sigma$ spanned by σ. A point in $\mathrm{relint}(\sigma)$ is given by a positive linear combination of a maximal subset of linear independent vectors of the generating set of σ.

Exercise Let $\tau \leq \sigma$ be a face. Then there is $u \in \mathrm{relint}(\tau^\perp \cap \sigma^\vee) \cap M$ s.t. $\tau = \sigma \cap u^\perp$.

Back to algebraic geometry

Let $T=(k^*)^d$ be a torus, and let $M = \mathrm{Hom}(T, k^*)$ be its character lattice. Let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice, and let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Then N is the dual vector space to M.

From now on we assume σ is a pointed rational polyhedral cone.

We let $S_\sigma := \sigma^\vee \cap M$ be the monoid associated to σ.

Fact from convex geometry (Gordan's Lemma): S_σ is finitely generated.

So $k[S_\sigma]$ is finitely generated. Also note that $S_\sigma \hookrightarrow M$, so $k[S_\sigma] \hookrightarrow k[M] = k[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$, so $k[S_\sigma]$ is a domain.

Definition: We let $U_\sigma := \mathrm{Spec}(k[S_\sigma])$ be the (affine) toric variety associated to σ.

Example If $\sigma = \langle e_1, \dots, e_d \rangle$, then $k[S_\sigma] = k[t_1, \dots, t_d]$ so $U_\sigma \cong \mathbb{A}^d$

Exercise If S_σ is generated by $u_1, \dots, u_t$, so we have

$$0 \to I \hookrightarrow k[y_1, \dots, y_t] \twoheadrightarrow k[\chi^{u_1}, \dots, \chi^{u_t}] = S_\sigma \to 0.$$

show that I is gen by binomials of the form $y_1^{a_1} \cdots y_t^{a_t} - y_1^{b_1} \cdots y_t^{b_t}$, where $a_1, \dots, a_t, b_1, \dots, b_t$ are nonneg integers satisfying $a_1 u_1 + \dots + a_t u_t = b_1 u_1 + \dots + b_t u_t$.

Note: $U_\sigma =$

Pick a standard basis $\langle e_1, \dots, e_d \rangle$ for N.

Proposition U_σ is smooth if and only if $S_\sigma \cong \mathbb{N}^r \times \mathbb{Z}^{d-r}$ for some r

if and only if $\sigma = \langle e_1, \dots, e_r \rangle$ for some r.

In particular, an affine smooth toric variety is isomorphic to $\mathbb{A}^r \times (k^*)^{d-r}$.

Proof omitted.

Exercise Let $\sigma = \langle e_1, v \rangle$. For which v is σ smooth? Classify all smooth affine toric surfaces.

We will restrict ourselves to smooth toric varieties.

<u>Fans and toric varieties</u>

A <u>fan</u> in N is a collection of prpc cones Δ s.t.

(i) every face of a cone in Δ is a cone in Δ

(ii) the intersection of any two cones in Δ is a face of both.

To each cone σ is associated an affine toric variety U_σ. These glue together as follows. Let $\tau \leq \sigma$. Then $\sigma^\vee \subseteq \tau^\vee$, so $k[S_\sigma] \to k[S_\tau]$ induces a morphism $U_\tau \to U_\sigma$.

<u>Lemma</u> Let $\tau \leqslant \sigma$ be a face, and $u \in \text{relint}(\tau^\perp \cap \sigma^\vee) \cap M$ with $\tau = \sigma \cap u^\perp$. Then $S_\tau = S_\sigma + \mathbb{N}(-u)$. And $U_\tau \subseteq U_\sigma$ is a principal open subset defined by $\{\chi^u \neq 0\}$.

<u>Proof</u> As $\sigma^\vee \subseteq \tau^\vee$ and $-u \in \tau^\perp \subseteq \tau^\vee$, "$\supseteq$" follows.

For the other inclusion, let $w \in S_\tau$. Since $u \in \text{relint}(\sigma^\vee \cap \tau^\perp)$, we have $\langle u, v_i \rangle > 0$ for all generators of σ that are not in τ. So there is $p \in \mathbb{N}$ with $w + pu \in \sigma^\vee$.

So $k[S_\tau] = k[S_\sigma][\chi^{-u}]$, i.e. $U_\tau = (U_\sigma)_{\chi^u}$.

Now if σ, σ' are cones in Δ and $\tau = \sigma \cap \sigma'$, then we can glue U_σ and $U_{\sigma'}$ along U_τ:

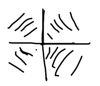
$$\begin{tikzcd} U_\sigma & & U_{\sigma'} \\ & U_\tau \arrow[ul] \arrow[ur, hook] & \end{tikzcd}$$

<u>Exercise</u>: These glueings are compatible.

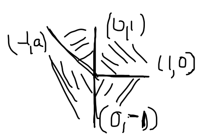
The resulting variety is denoted $X(\Delta)$.

<u>Fact</u> $X(\Delta)$ is separated (proof omitted).

<u>Exercise</u> 1) Let $\Delta =$  Then $X(\Delta) = k \times \mathbb{P}^1$

2) Let $\Delta =$  Then $X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$

3) Let $\Delta =$  Then $X(\Delta) = Bl_p \mathbb{A}^2$

4) Let $\Delta =$ 

Then $X_\Delta = \mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$.

(If you have not met $\mathbb{F}_a$ before, you can take this as the definition.)

<u>Fact</u>: $X(\Delta)$ is complete $\iff$ Δ is complete, i.e., $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.

A <u>lattice polytope</u> is the convex hull of finitely many lattice points.

To a polytope in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ is associated a fan Δ in $N_{\mathbb{R}}$ as follows.

For each face F of P, let $\sigma_F := \{ v \in N_{\mathbb{R}} \mid \langle u' - u, v \rangle \geq 0 \quad \forall u' \in P, u \in F \}$.

<u>Fact</u> $\Delta_P = \{ \sigma_F \mid F \text{ a face of } P \}$ is a fan

<u>Fact</u> $X(\Delta)$ is projective if and only if $\Delta = \Delta_P$ for some lattice polytope P.

<u>Exercise</u> Let $P = \operatorname{conv}\langle 0, 1 \rangle$. Find Δ_P and $X(\Delta_P)$.

Let $P = \operatorname{conv}\langle (0,0), (1,0), (0,1) \rangle$. Find Δ_P and $X(\Delta_P)$.

Let $P = \operatorname{conv}\langle (0,0), (1,0), (0,1), (1,1) \rangle$. Find Δ_P and $X(\Delta_P)$.

The torus action and torus orbits

points in U_σ $\longleftrightarrow$ maximal ideals $m_x = \{ f \in k[S_\sigma] \mid f(x) = 0 \} \subseteq k[S_\sigma]$

$\longleftrightarrow$ k-algebra morphisms $k[S_\sigma] \to k[S_\sigma]/m_x \cong k$

$\longleftrightarrow$ monoid morphisms $S_\sigma \to (k, \cdot)$

In particular $t \in T \longleftrightarrow$ monoid morphism $M \to k$

Then for $t: M \to k$, $x: S_\sigma \to k$, we have

$(t \cdot x): S_\sigma \to k, \quad u \mapsto t(u)\, x(u)$.

Alternatively: If S_σ is generated by $u_1, \dots, u_t$, then $U_\sigma \hookrightarrow \mathbb{A}^t$, and $T_d \hookrightarrow T_t$ by $\vec{t} = (t_1, \dots, t_d) \mapsto (\vec{t}^{\,u_1}, \dots, \vec{t}^{\,u_t})$, and $T_d \circlearrowright \mathbb{A}^t$ via the action of T_t on $\mathbb{A}^t$. (Check: $(y_1, \dots, y_t) \in U_\sigma \Rightarrow \vec{t} \cdot (y_1, \dots, y_t) \in U_\sigma$).

U_τ has a special point:

$$x_\tau : S_\tau \to k, \quad u \mapsto \begin{cases} 1 & \text{if } u \in \tau^\perp \\ 0 & \text{otherwise} \end{cases}$$

When σ is maximal, then $\sigma^\vee$ is pointed and $\sigma^\perp = \{0\}$, then x_σ is the unique torus fixed point in U_σ and

$$m_{x_\sigma} = \bigoplus_{u \in S_\sigma \setminus \{0\}} k\chi^u \qquad (\chi^u(x_\sigma) = 0 \Leftrightarrow x_\sigma(u) = 0)$$

Def: let $O_\tau = T \cdot x_\tau$ and $V(\tau) = \overline{O_\tau}$.

$V(\tau)$ can be described as follows: Fix $\tau \in \Delta$, let $N(\tau) = N / \langle \tau \cap N \rangle_{\mathbb{R}}$. Then $M(\tau) = N(\tau)^\vee = M \cap \tau^\perp$. To τ is associated a fan in $N(\tau)$, denoted $\mathrm{Star}(\tau)$.

The cones in $\mathrm{Star}(\tau)$ are the images of the cones σ containing τ in $N(\tau)$.

Then $V(\tau) := V(\mathrm{Star}(\tau))$. Note: $\dim V(\tau) = \operatorname{codim} \tau$

Fact There is an order-reversing correspondence between cones τ of Δ and T-invariant subvarieties $V(\tau)$.

Prop: i) $U_\sigma = \coprod_{\tau < \sigma} O_\tau$ ii) $V(\tau) = \coprod_{\gamma \geq \tau} O_\gamma$

Proof (i) is exercise. Show that if $x \in U_\sigma$, then $x^{-1}(k^*) = \sigma^\vee \cap \tau^\perp \cap M$ for a face $\tau \leq \sigma$. Use the fact that a convex set $C \subseteq \sigma$ is a face if it satisfies the property $v + v' \in C$ only if $v, v' \in C$, where $v, v' \in \sigma$.
ii) omitted.

Exercise: Let Δ be a complete fan, τ a codim-1 face in Δ. Then $V(\tau) \cong \mathbb{P}^1$.

Divisors on toric varieties

Let $X = X(\Delta)$ be a smooth toric variety.
Since X is smooth, we have $Cl(X) = Pic(X)$, where

$$Pic(X) = \frac{\text{Group of Cartier divisors on } X}{\text{linear equivalence}}$$

Let $\Delta(1) = \{\rho_1, \dots, \rho_n\}$ denote the set of rays in Δ. To ρ_i is associated a T-invariant divisor, $D_i = V(\rho_i)$. They form a subgroup $Div_T(X)$ of all divisors.

Let $u \in M$, and χ^u be the associated character, a rational function on X.

For a ray g_i, let v_i denote the primitive vector spanning g_i.

<u>Lemma</u> A Let $u \in M$. Then
$$\mathrm{div}(\chi^u) = \sum_{i=1}^{n} \langle u, v_i \rangle D_i .$$

<u>Proof</u> We can replace X by $U_{g_1} \cup \dots \cup U_{g_r}$ as the complement has codim 2. Each $U_{g_i} \simeq \mathrm{Spec}(k[x_1]) \times (k^*)^{d-1}$, $U_{g_i} \cap D_i = 0 \times (k^*)^{d-1}$ and $U_{g_i} \cap D_j = \emptyset$. We can choose a basis $e_1, \dots, e_d$ of N s.t. $v_i = e_1$. Then $\chi^u|_{U_{g_i}} = x_1^{u_1} \dots x_d^{u_d}\Big|_{U_{g_i}} = x_1^{\langle u, v_1 \rangle} x_2^{u_2} \dots x_d^{u_d}\Big|_{U_{g_i}}$ vanishes to order $\langle u, v_i \rangle$ along D_i. $\square$

A <u>Cartier T-divisor</u> is a Cartier divisor that corresponds to a divisor in $\mathrm{Div}_T(X)$.

<u>Fact from Representation Theory</u>: If $V \subseteq k[M]$ is T-invariant submodule then V is generated by χ^u for $u \in M$.

<u>Lemma B</u> Let σ be a pointed cone in Δ. Then for every T-divisor D on U_σ there is $u \in M$ s.t.
$$D = \mathrm{div}(\chi^u).$$

Proof: Note that $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ is supported on the complement of T, so if $f \in \mathcal{O}(D)$, then f is a regular function on T. This means that $f \in k[M]$. T acts on $K(X)$ by $(t \cdot f)(x) := f(t^{-1}x)$. Since D is T-invariant, for $f \in \mathcal{O}(D)$ we have $tf \in \mathcal{O}(D)$, so $H^0(U_\sigma, \mathcal{O}(D)) \subseteq k[M]$ is a T-invariant submodule. (If $x \in U$, then $\mathrm{ord}_x f = \mathrm{ord}_{tx}(tf)$ since $(tf)(tx) = f(x)$.)

By the fact, $H^0(U_\sigma, \mathcal{O}(D))$ is is M-graded, i.e., of the form $\bigoplus_u \mathbb{C}\chi^u$, where u ranges over a suitable subset of M. We claim there is u s.t.

$$H^0(U_\sigma, \mathcal{O}(D)) = \chi^{-u} \cdot k[U_\sigma].$$

Step 1 $\dim \sigma = d$. As D is Cartier, in a neighbourhood $\bar{U}$ of x_σ, $H^0(\bar{U}, \mathcal{O}(D))$ has rank 1. So $\dim \left(H^0(U_\sigma, \mathcal{O}(D)) / m_{x_\sigma} H^0(U_\sigma, \mathcal{O}(D)) \right) = 1$ and this is M-graded. So it must be generated by χ^{-u} for a unique $u \in M$. By graded Nakayama, χ^{-u} also generates $H^0(U_\sigma, \mathcal{O}(D))$, so $H^0(U_\sigma, \mathcal{O}(D)) = \chi^{-u} k[S_\sigma]$.

Step 2 $\dim \sigma < d$. Then $U_\sigma = U_{\sigma'} \times (k^*)^{d - \dim(\sigma)}$, where σ' is the cone in $N \cap \langle \sigma \rangle$ with the same rays as σ. Let $D' = j^* D$, where $j : U_{\sigma'} \hookrightarrow U_\sigma$. By step 1, there is u' in the dual lattice to $N \cap \langle \sigma \rangle$, $M / M \cap \sigma^\perp$, with $D' = \mathrm{div}(\chi^{u'})$. Then any element u in M mapping to $u' \in M / M \cap \sigma^\perp$ satisfies $\mathrm{div}\, \chi^u = D$.

Now let $X = X(\Delta)$ be a smooth toric variety and let $D = \sum a_i D_i$ be a T-Cartier divisor. Let σ be a maximal cone. Define $u(\sigma) \in M$ by $\langle u(\sigma), v_\rho \rangle = -a_\rho$ for all $\rho \in \sigma$. Then $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})$ and $\Gamma(U_\sigma, \mathcal{O}(D)) = \chi^{u(\sigma)} \cdot k[U_\sigma]$. So D determines a set $\{u(\sigma)\}_{\sigma \text{ maximal cone in } \Delta}$.

Conversely, a set $\{u(\sigma)\}_{\sigma \text{ max'l cone}}$ gives rise to a Cartier divisor if and only if for every pair σ, σ' $\mathrm{div}(\chi^{-u(\sigma)})\big|_{U_\sigma \cap U_{\sigma'}} = \mathrm{div}(\chi^{-u(\sigma')})\big|_{U_\sigma \cap U_{\sigma'}}$

The torus invariant divisors meeting $U_\sigma \cap U_{\sigma'}$ correspond to the rays contained in $\tau = \sigma \cap \sigma'$, and so we need $\langle u(\sigma), v_\rho \rangle = \langle u(\sigma'), v_\rho \rangle$ for all rays ρ of τ, which implies $u(\sigma)\big|_\tau = u(\sigma')\big|_\tau$, i.e. $u(\sigma) - u(\sigma') \in \tau^\perp \cap M$. So we get

$$\{\text{T-Cartier divisors}\} = \mathrm{Ker}\Big(\bigoplus_{\text{max'l cones}} M \longrightarrow \bigoplus_{\sigma, \sigma' \text{ max'l cone}} M / M \cap (\sigma \cap \sigma')^\perp \Big)$$

Let D be a T-Cartier divisor. By Lemma A, to each cone $\sigma \in \Delta$ is associated $u(\sigma) \in M$ s.t. $D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})$. We let

$$\Psi_D = \{u(\sigma)\}_{\sigma \text{ maximal cone of } \Delta} \subseteq M$$

and we let

$$\psi_D : |\Delta| \to \mathbb{R}, \quad v \mapsto \langle u(\sigma), v \rangle \text{ if } v \in \sigma.$$

Note that ψ_D is well-defined since if $\sigma \cap \sigma' = \tau$, then $u(\sigma)\big|_\tau = u(\sigma')\big|_\tau$.

<u>Exercise</u>: Let $X = \mathrm{Bl}_{pt}\, \mathbb{P}^2$. Find ψ_D for $D = E$, $D = H$, $D = E + H$

We are now ready to describe $\mathrm{Pic}(X)$:

<u>Proposition</u> Let $X = X(\Delta)$ be a smooth toric variety. Then there is the following exact sequence:

$$0 \to M \to \bigoplus_{i=1}^{n} \mathbb{Z} D_i \to \mathrm{Pic}(X) \to 0,$$

where the first map is given by $u \mapsto \mathrm{div}(\chi^u)$, and the second map by $D \mapsto [D]$.

<u>Proof</u>: Note that $X \setminus \bigcup D_i = T_N$ which is affine with coordinate ring $k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, a UFD. So every divisor on T_N is principal.

We first show that every divisor is linearly equivalent to a T-divisor. Indeed, let D be a divisor on X. Let $j: T_N \hookrightarrow X$ be the inclusion. Then j^*D is a divisor on T_N, so $j^*D = \mathrm{div}(f)$ for some $f \in k(T_N) = k(X)$. So $D - \mathrm{div}(f)$ is supported on the complement of T_N, which is $\bigcup D_i$. So in $\mathrm{Pic}(X)$ we have $[D] = [\sum a_i D_i]$, and we get a surjection $\bigoplus \mathbb{Z} D_i \to \mathrm{Pic}(X)$.

To show exactness in the middle, first note that $u \mapsto \mathrm{div}(\chi^u) \mapsto 0$ in $\mathrm{Pic}(X)$. Let $D = \sum a_i D_i$ be T-invariant and suppose $[D] = 0$ in $\mathrm{Pic}(X)$, so $D = \mathrm{div}(f)$ for some $f \in k(X)$. By Lemma A, we have $D|_{T_N} = \mathrm{div}(\chi^u)$ for some $u \in M$. Thus $f = \lambda \chi^u$ for some $\lambda \in k^*$. Since the v_i span $N_{\mathbb{R}}$, u is determined uniquely by f. Thus exactness follows. $\square$

<u>Exercise</u> Find Pic of $\mathrm{Bl}_p \mathbb{P}^2$.

Positivity of toric divisors

Suppose X is complete.

To a Cartier divisor with global sections is associated a map to projective space.

We let

$$H^0(X, \mathcal{O}(D)) = \{ f \in K(X) \mid \operatorname{div}(f) + D \geq 0 \}$$

Then $\dim H^0(X, \mathcal{O}(D)) < \infty$, so pick a basis $f_0, \dots, f_N$. We let

$$\varphi_D : X \dashrightarrow \mathbb{P}^N, \quad x \mapsto \text{"}[f_0(x) : \dots : f_N(x)]\text{"}$$

Positivity question: When is this map a morphism? An embedding? Satisfy other wonderful properties?

How is φ_D defined? Let $x \in X$. As D is Cartier, there is a neighborhood U of x and $f_{D,x} \in K(X)$ s.t. $D|_{U_x} = \operatorname{div}(f_{D,x}^{-1})$. Then $H^0(U_x, \mathcal{O}(D)) = f_{D,x}\, k[U_x]$.

$$\Big[\operatorname{div}(f|_{U_x}) + D|_{U_x} \geq 0 \iff \operatorname{div}(f|_{U_x}) + \operatorname{div}(f_{D,x}^{-1}|_{U_x}) \geq 0$$
$$\iff \operatorname{div}(f \cdot f_{D,x}^{-1}|_{U_x}) \geq 0$$
$$\iff f \cdot f_{D,x}^{-1}|_{U_x} \in k[U_x]$$
$$\iff f|_{U_x} \in f_{D,x}|_{U_x} k[U_x] \Big]$$

Now if $f \in H^0(X, \mathcal{O}(D))$, then $f|_{U_x} \in H^0(U_x, \mathcal{O}(D))$, and so $f \cdot f_{D,x}^{-1}|_{U_x} \in k[U_x]$.

We define $\varphi_D|_{U_x} : U_x \dashrightarrow \mathbb{P}^N$, $x \mapsto [f_0(x) f_{D,x}^{-1}(x) : \dots : f_N(x) f_{D,x}^{-1}(x)]$.

Check that this is well-defined

The toric case:

<u>Definition</u> Let $D=\sum a_i D_i$ be a divisor. To D is associated a polytope
$$P_D=\{u\in M \mid \langle u,v_i\rangle \geq -a_i \ \forall i\}.$$

<u>Proposition</u>: Let $D=\sum a_i D_i$ be a divisor on X. Then
$$H^0(X,\mathcal{O}(D))=\bigoplus_{u\in P_D\cap M} k\chi^u$$

<u>Proof</u>: Since D is T-invariant, and X is T-invariant, $H^0(X,\mathcal{O}(D))$ is a T-invariant subspace of $k[M]$. By Fact from rep. theory it is generated by $\{\chi^u, u\in M\}$.
But $\chi^u\in H^0(X,\mathcal{O}(D)) \iff \operatorname{div}(\chi^u)+D\geq 0 \underset{\text{Lemma A}}{\iff} \langle u,v_i\rangle + a_i \geq 0 \ \forall i.$ $\square$

<u>Exercise</u> Find P_D when a) $X=\mathbb{P}^1$, $D=D_\rho$ ρ any ray in fan of $\mathbb{P}^1$.

b) $X=\mathbb{P}^1\times\mathbb{P}^1$, $D=D_{\langle e_1\rangle}+D_{\langle e_2\rangle}$

c) $X=\mathbb{P}^2$, $D=mD_{\langle e_1\rangle}$. What is the associated embedding?

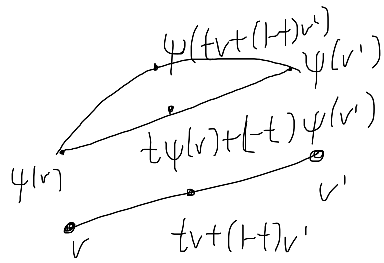
<u>Exercise</u> Show that $P_{mD}=mP_D$ for all positive integers m. If $u\in M$, then $P_{D+\operatorname{div}(\chi^u)}=P_D-u$. Moreover, if D,D' are Weil-divisors, then $P_D+P_E\subseteq P_{D+E}$.

Let D be a Cartier T-divisor on X, and let $D|_{U_\sigma}=\operatorname{div}(\chi^{-u(\sigma)})$.
Let $P_D\cap M=\{u_0,\dots,u_N\}$. Then
$$\varphi_D|_{U_\sigma}: U_\sigma \to \mathbb{P}^N,\quad x\mapsto [\chi^{u_0-u(\sigma)}:\dots:\chi^{u_N-u(\sigma)}]$$

<u>When is φ_D a morphism?</u> (D "basepoint free", $\mathcal{O}(D)$ globally generated)

A function ψ defined on a subset of $N_{\mathbb{R}}$ is called <u>concave</u> if for all v, v' and $0 \le t \le 1$ we have

$$\psi(tv + (1-t)v') \ge t\psi(v) + (1-t)\psi(v')$$



whenever $v, v', tv+(1-t)v'$ are in the domain of ψ.

Now let Δ be a fan in $N_{\mathbb{R}}$ and ψ be a function that is piecewise linear on Δ, i.e., for every maximal cone σ there is $u(\sigma) \in M_{\mathbb{R}}$ $\psi|_\sigma = u(\sigma)$.

From now on we assume $|\Delta|$ is complete.

<u>Lemma</u> C ψ is concave if and only if for every maximal cone σ and for every $v \in |\Delta|$ we have

$$\langle u(\sigma), v\rangle \ge \psi(v). \quad (*)$$

<u>Proof</u> First assume that $(*)$ holds for every maximal cone σ. Let $v, v' \in |\Delta|$ and $0 \le t \le 1$. Let σ be a maximal cone in Δ s.t. $tv + (1-t)v' \in \sigma$. Then

$$\psi(tv+(1-t)v') = \langle u(\sigma), tv + (1-t)v'\rangle = t\langle u(\sigma), v\rangle + (1-t)\langle u(\sigma), v'\rangle$$
$$\underset{(*)}{\ge} t\psi(v) + (1-t)\psi(v').$$

Conversely, assume ψ is concave. Let $v \in |\Delta|$ and let σ be a maximal cone. Let w be in the interior of σ. Then there is $0 < t < 1$ s.t. $tw + (1-t)v \in \sigma$.

Then

$$t\langle u(\sigma), w\rangle + (1-t)\langle u(\sigma), v\rangle = \langle u(\sigma), tw + (1-t)v\rangle$$
$$= \psi(tw + (1-t)v)$$
$$\geq t\,\psi(w) + (1-t)\,\psi(v)$$
$$= t\langle u(\sigma), w\rangle + (1-t)\,\psi(v).$$

So $\langle u(\sigma), v\rangle \geq \psi(v)$. $\square$

<u>Proposition</u> $\mathcal{O}(D)$ is globally generated if and only if ψ_D is concave.

<u>Proof</u> Let σ be a maximal cone. Then $U_\sigma \cong \mathbb{A}^d$ and $k[U_\sigma] \cong k[x_1, \dots, x_n]$. Moreover, $\Gamma(U_\sigma, \mathcal{O}(D)) = \chi^{u(\sigma)} k[x_1, \dots, x_n]$ and the isomorphism

$\Gamma(U_\sigma, \mathcal{O}(D)) \longrightarrow k[U_\sigma]$ is given by multiplication with $\chi^{-u(\sigma)}$.

Note that the only non-vanishing T-invariant function on $0 \in \mathbb{A}^n \cong U_\sigma$ is the constant function λ which is nonvanishing on all of $\mathbb{A}^n$. Note that the section of $\mathcal{O}(D)(U_\sigma)$ corresponding to the constant function λ under the above isomorphism is $\lambda \chi^{u(\sigma)}$. This extends to a global section if and only if $\mathrm{div}(\chi^{u(\sigma)}) + D \geq 0$, i.e., if $\langle u(\sigma), v_\rho\rangle \geq -a_\rho$ for all rays ρ in Δ. Thus $\mathcal{O}(D)$ is globally generated if and only if for all maximal cones σ of Δ, we have

$$\langle u(\sigma), v_\rho\rangle \geq -a_\rho = \psi(v_\rho)$$

for all ρ. This is equivalent to $\langle u(\sigma), v\rangle \geq \psi(v)$ for all $v \in |\Delta|$.

Indeed, if $v \in |\Delta|$, let σ' be a maximal cone containing v. Then $v = \sum_{\varrho \in \sigma'(1)} \lambda_\varrho v_\varrho$ with $\lambda_\varrho \geq 0$. So we get

$$\langle u(\sigma), v\rangle = \sum \lambda_\varrho \langle u(\sigma), v_\varrho\rangle \geq \sum \lambda_\varrho \psi(v_\varrho) \underset{\uparrow}{=} \psi(v)$$

ψ is liner on σ'

Now the claim follows from Lemma C.

<u>Note</u> The proof implies that if D is globally generated, then $u(\sigma) \in P_D$

<u>Exercise</u> Modify the proof for singular toric varieties.

<u>Fact</u> Let D be a T-Cartier divisor s.t. $\mathcal{O}(D)$ is globally generated. Then $\{ u(\sigma) \mid D|_{U_\sigma} = \operatorname{div}(\chi^{-u(\sigma)}) \}_{\sigma \text{ max'l cone}}$ is the set of vertices of P_D.

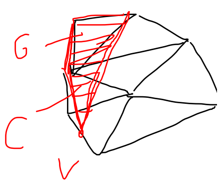
When is φ_D an embedding? ($D/\mathcal{O}(D)$ very ample)

We say that ψ_D is strictly concave if ψ_D is concave and the inequality
$$\langle u(\sigma), v\rangle \geq \psi_D(v) \text{ is strict unless } v \in \sigma.$$

Lemma D ψ_D is strictly concave if and only if ψ_D is concave and for every pair of maximal cones σ, σ', we have $u(\sigma) \neq u(\sigma')$.

Proof "$\Rightarrow$" ψ_D is concave by assumption. If $u(\sigma) = u(\sigma')$ for some $\sigma \neq \sigma'$, then for $v \in \sigma' \setminus \sigma$ we have $\psi_D(v) = \langle u(\sigma'), v\rangle = \langle u(\sigma), v\rangle > \psi_D(v)$, a contradiction.

"$\Leftarrow$" Let σ be a maximal cone and let $v \in N_{\mathbb{R}} = |\Sigma| \setminus \sigma$. We need to show that $\langle u(\sigma), v\rangle > \psi_D(v)$. Suppose equality holds. Let $C = \operatorname{conv}\langle \sigma, v\rangle$ be the convex hull of σ and v. We claim that $\psi_D|_C = u(\sigma)$. Indeed, let $\tilde{v} \in C$. Then $\tilde{v} = tw + (1-t)v$ for some $w \in \sigma$.



$$\text{Then } \langle u(\sigma), \tilde{v}\rangle \underset{\substack{\uparrow \\ \text{Lemma C}}}{\geq} \psi_D(\tilde{v}) \underset{\substack{\uparrow \\ \text{Concavity} \\ \text{of } \psi_D}}{\geq} t\,\psi_D(w) + (1-t)\,\psi_D(v)$$

$$= t\langle u(\sigma), w\rangle + (1-t)\langle u(\sigma), v\rangle = \langle u(\sigma), \tilde{v}\rangle$$

Since σ is full-dimensional, C must contain an open subset of the interior of some maximal cone σ' containing v. Since $\psi_D|_{\sigma'}$ is linear, its values are determined by its values on an open subset of σ', so it follows that $u(\sigma) = u(\sigma')$, a contradiction.

Proposition $\mathcal{O}(D)$ is very ample if and only if φ_D is strictly concave.

Proof "$\Leftarrow$" $\mathcal{O}(D)$ is globally generated by previous Prop. Let

$$\varphi_D : X \to \mathbb{P}(H^0(X, \mathcal{O}(D))) = \mathbb{P}^N$$

be the induced morphism. Let $P_D = \{u_0, \dots, u_N\}$, then φ_D is given by the global sections $\chi^{u_0}, \dots, \chi^{u_N}$. Let $T_0, \dots, T_N$ be the corresponding coordinates on $\mathbb{P}^N$. Let σ, σ' be a maximal cones of Δ and let j, j' be s.t. $u(\sigma) = u_j$ and $u(\sigma') = u_{j'}$. Then since $\chi^{u(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ, we have $\chi^{u(\sigma)}(x) \neq 0 \ \forall x \in U_\sigma$, so $\varphi_D(U_\sigma) \subseteq \{T_j \neq 0\} \subseteq \mathbb{P}^N$.

Claim: $\varphi_D^{-1}(\{T_j \neq 0\}) \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$. Indeed, if $x \in U_{\sigma \cap \sigma'} = U_\sigma \cap U_{\sigma'}$, $x \in U_\sigma$, so $\varphi_D(x) \in \{T_j \neq 0\}$ and so $x \in \varphi_D^{-1}(\{T_j \neq 0\})$, so $x \in \varphi_D^{-1}(\{T_j \neq 0\}) \cap U_{\sigma'}$. Conversely, let $x \in \varphi_D^{-1}(\{T_j \neq 0\}) \cap U_{\sigma'}$.
Note that

$$\varphi_D\big|_{U_\sigma} = \Big(\chi^{u_0 - u(\sigma)}, \dots, \underset{\substack{\shortparallel \\ 1}}{\chi^{u_j - u(\sigma)}}, \dots, \chi^{\overset{\substack{u(\sigma') \\ \shortparallel}}{u_{j'}} - u(\sigma)}, \dots, \chi^{u_N - u(\sigma)}\Big)$$

Let O_τ be the orbit containing x. As $U_\sigma = \bigcup_{\tau < \sigma} O_\tau$, it suffices to show that $\tau < \sigma$. Let ρ be a ray of τ. Note that $D_\rho = V(\rho) = \bigcup_{\tau \ni \rho} O_\tau$.
Now $x \in U_{\sigma'}$, so $\varphi_D(x) \subseteq \{T_{j'} \neq 0\}$, so $\chi^{u(\sigma') - u(\sigma)}(x) \neq 0$, so
$0 = \operatorname{ord}_{D_\rho} \chi^{u(\sigma') - u(\sigma)} = \langle u(\sigma') - u(\sigma), v_\rho \rangle$. By strict concavity of φ_D, and Lemma D, $\langle u(\sigma) - u(\sigma'), v_\rho \rangle = 0 \Leftarrow\Rightarrow \rho \in \sigma$. Thus $\tau \in \sigma$, and the claim follows.

The Claim implies that $U_\sigma = f^{-1}(V_j)$

$$\left[\; U_\sigma = \left(\bigcup_{\sigma'} U_{\sigma'}\right) \cap U_\sigma = \bigcup_{\sigma'} U_{\sigma'} \cap U_\sigma = \bigcup_{\sigma'} f^{-1}(V_j) \cap U_{\sigma'} \right.$$

$$\left. = f^{-1}(V_j) \cap \left(\bigcup_{\sigma'} U_{\sigma'}\right) = f^{-1}(V_j) \;\right]$$

So it suffices to show that $U_\sigma \hookrightarrow V_j$ is an embedding.

WLOG we may assume that $\sigma = \langle e_1, \dots, e_d \rangle$, where $e_1, \dots, e_d$ are a basis of N. Let $e_1^*, \dots, e_d^*$ be the dual basis. Then $\sigma^\vee = \langle e_1^*, \dots, e_d^* \rangle$

On the ring level, $\varphi_D|_{U_\sigma}$ is induced by

$$k\left[\frac{T_0}{T_j}, \dots, \widehat{\frac{T_j}{T_j}}, \dots, \frac{T_N}{T_j}\right] \to k[\sigma^\vee \cap M] \cong k[x_1, \dots, x_d]$$

$$\frac{T_i}{T_j} \longmapsto \chi^{u_i - u(\sigma)}$$

and $\varphi_D|_{U_\sigma}$ is an immersion if and only if this morphism of rings is surjective, i.e., for all $1 \le l \le d$ there is l s.t. $u_l - u(\sigma) = e_l^*$

Let σ_l be the maximal cone that shares the facet $\langle e_1, \dots, \hat{e}_l, \dots, e_d \rangle$ with σ.

We have

$$\langle u(\sigma_l), e_j \rangle = \psi_D(e_j) = \langle u(\sigma), e_j \rangle \quad \text{for } j \neq l.$$

So $u(\sigma_l) - u(\sigma) \in \langle e_1, \dots, \hat{e}_l, \dots, e_d \rangle^\perp = \langle e_l^* \rangle$.

Since ψ_D is strictly concave, we also have

$$\langle u(\sigma_l), e_l \rangle > \psi_D(e_l) = \langle u(\sigma), e_l \rangle.$$

So we have
$$u(\sigma_e) - u(\sigma) = m e_e^*$$
for some positive integer m. Since P_D is convex, it follows that
$$u_e := u(\sigma) + \frac{1}{m}\left(u(\sigma_e) - u(\sigma)\right) = \left(1 - \frac{1}{m}\right) u(\sigma) + \frac{1}{m} u(\sigma_e) \in P_D$$
Then $u_e - u(\sigma) = e_e^*$.

" $\Leftarrow$ " Now assume that $\mathcal{O}(D)$ is very ample. Then $\mathcal{O}(D)$ is globally generated, so φ_D is concave by Proposition on global gen. By lemma D, it suffices to show that for any pair of maximal cones σ, σ', we have $u(\sigma) \neq u(\sigma')$.

Suppose $u(\sigma) = u(\sigma')$ for some such pair. Then
$$\Psi_D\big|_{U_\sigma} = [\chi^{u_1 - u(\sigma)} : \dots : \chi^{u_N - u(\sigma)}], \quad \text{and}$$
$$\chi^{u_i - u(\sigma)}(x_\sigma) = \begin{cases} 1 & \text{if } u_i = u(\sigma) \\ 0 & \text{if } u_i \neq u(\sigma) \end{cases} \quad \left(\begin{array}{l}\text{recall that } U_\sigma \cong \mathbb{A}^d \\ \text{and } \quad x_\sigma \leftrightarrow 0\end{array}\right)$$

So $\Psi_D\big|_{U_\sigma}(x_\sigma) = [T_0 : \dots : T_N]$, where $T_i = \begin{cases} 1 & \text{if } u_i = u(\sigma) \\ 0 & \text{if } u_i \neq u(\sigma) \end{cases}$

In particular, for $\sigma' \in \{\sigma' \text{ max'l cone} \mid u(\sigma') = u(\sigma)\}$, we have $\Psi_D(x_{\sigma'}) = \Psi_D(x_\sigma)$.
As Ψ_D is injective, it follows that $|\{\sigma' \text{ max'l} \mid u(\sigma') = u(\sigma)\}| = 1$, a contradiction. □

<u>Prop</u> Let X be a complete toric variety (possibly singular).

1) D is ample $\Leftrightarrow$ φ_D is strictly concave

2) D is very ample if for all maximal cones σ, $P_D \cap M - u(\sigma)$ generates S_σ.

3) If X is smooth, every ample divisor is very ample.

Syzygies of toric varieties

Let D be very ample and let $R(D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}(mD))$.
R is a graded $S := \mathrm{Sym}^\bullet H^0(X, \mathcal{O}(D)) = k[X_0, \dots, X_N]$ - module.
So it admits a minimal free graded resolution

$$0 \to E_N \to \dots \qquad \dots \to E_2 \to E_1 \to E_0 \to R(D) \to 0$$

D satisfies N_0 if $E_0 \simeq S \Leftrightarrow S \twoheadrightarrow R \Leftrightarrow R$ gen. in deg 1

N_1 if also $E_1 \simeq \bigoplus S(-2) \Leftrightarrow$ also $I_{X/\mathbb{P}^N}$ generated by eqns in degree 2

N_2 if also $E_2 \simeq \bigoplus S(-3) \Leftrightarrow$ first syzygies are linear.

[$(F_1, \dots, F_r) = I$ minimal generating set, then a first syzygy is minimal generator of module $(G_1, \dots, G_r) : G_1F_1 + \dots + G_rF_r = 0$]

N_p if N_0 & $E_i \simeq \bigoplus S(-i-1)$ for $1 \leq i \leq p$.

Theorem (HSS) X toric, then $(n-1+p)D$ satisfies N_p.

Remark: The minimal free resolutions are not understood for $\mathbb{P}^2$!
We know that $\mathcal{O}_{\mathbb{P}^2}(d)$ satisfies N_{3d-3}, but not the exact dimensions of the graded pieces.
For $\mathbb{P}^{3/n}$ we don't even know whether $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies N_{3d-3}.