9
Bases

Certain subsets of the coordinate space $\mathbb{K}^m$ inherit the features of the ambient space. In this chapter, we familiarize ourselves with linear subspaces, examine the minimum number of scalars needed to specify a vector in such subspace, and analyze the coordinates that uniquely determine a vector with such a linear subspace.

9.0 Linear Subspaces

What are the natural collections of vectors? For a fixed a nonnegative integer m, consider the coordinate space $\mathbb{K}^m$.

9.0.0 Definition. A ***linear subspace*** in $\mathbb{K}^m$ is any nonempty subset V of $\mathbb{K}^m$ that is closed under taking linear combinations: for any finite collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in V and any scalars $c_1, c_2, \dots, c_n$ in $\mathbb{K}$, the linear combination $c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \cdots + c_n\,\vec{v}_n$ also lies in V.

Every linear subspace contains the zero vector because the empty sum (when $n = 0$) is additive identity.

9.0.1 Remarks.
- The largest linear subspace in $\mathbb{K}^m$ is the entire space.
- The smallest linear subspace is the zero subspace $\{\vec{\mathbf{0}}\}$ consisting of just the zero vector.
- For any vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{K}^m$, the set $\operatorname{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$ is a linear subspace.
- Lemma 5.1.1 shows that the kernel of a matrix is a linear subspace.

In addition to its kernel, there are two more canonical linear spaces associated to a matrix.

9.0.2 Definition. The ***column space*** of an $(m \times n)$-matrix $\mathbf{A}$ is the linear subspace of $\mathbb{K}^m$ spanned by the columns of $\mathbf{A}$. The ***row space*** of an $(m \times n)$-matrix $\mathbf{A}$ is the linear subspace of $\mathbb{K}^n$ spanned by the columns of $\mathbf{A}^\mathsf{T}$.

9.0.3 Proposition. *Two matrices have the same reduced row echelon form if and only if their row spaces are equal.*

Proof. It suffices to show that two matrices related by an elementary row operation have the same row space. Suppose that, for some matrices $\mathbf{A}$ and $\mathbf{B}$, there exists an elementary matrix $\mathbf{R}$ such that $\mathbf{R}\,\mathbf{A} = \mathbf{B}$. To demonstrate that the row spaces of $\mathbf{A}$ and $\mathbf{B}$ are equal, we prove containment in both directions.

$\subseteq$: The properties of the transpose [5.2.7] imply that $\mathbf{B}^\mathsf{T} = \mathbf{A}^\mathsf{T}\,\mathbf{R}^\mathsf{T}$ and the definition of matrix multiplication [7.0.0] implies that each row of $\mathbf{B}$ is a linear combination of the rows in $\mathbf{A}$. Hence, the row space of $\mathbf{B}$ is contained in the row space of $\mathbf{A}$.

$\supseteq$: Since elementary matrices are invertible and their inverses are also elementary matrices [8.0.2], we have $\mathbf{A} = \mathbf{R}^{-1}\,\mathbf{B}$ and $\mathbf{A}^\mathsf{T} = \mathbf{B}^\mathsf{T}\,(\mathbf{R}^{-1})^\mathsf{T}$. Again, we see that each row of $\mathbf{A}$ is a linear combination of the rows in $\mathbf{B}$, so the row space of $\mathbf{A}$ is contained in the row space of $\mathbf{A}$. □

9.0.4 Warning. Elementary row operations typically change the column space of a matrix.

Linear subspaces have preferred spanning sets.

9.0.5 Definition. A ***basis*** for a linear subspace V is a collection of vectors that form a linearly independent spanning set.

9.0.6 Remark. For any positive integer n, the columns of an invertible $(n \times n)$-matrix form a basis of $\mathbb{K}^n$, because the characterization of invertible matrices [8.0.3] shows that they are linearly independent and span $\mathbb{K}^n$. In particular, the columns $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ of the identity matrix form a basis.

9.0.7 Proposition. *For any matrix in reduced row echelon form, the columns containing the leading ones form a basis for its column space and the nonzero rows form a basis for its row space.*

Proof. For any matrix in in reduced row echelon form, the columns of a matrix containing the leading ones are linearly independent because they are a subset of the standard basis. Since the nonzero entries in a column only appear in a row containing a leading one, we see that any column lies in the span of the columns containing the leading entries. Hence, the columns containing the leading ones are a basis for the column space

Since a zero row is trivially a linear combination of the nonzero rows, these rows span the row space. As the leading ones are the unique nonzero entry in their column, the only linear combination of these rows that equals zero is the zero combination, so the nonzero rows are linearly independent. Thus, the nonzero rows in the matrix form a basis for the row space. □

9.0.8 Problem. Find a basis for the kernel of the matrix

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. The row reduction algorithm [4.2.0] gives

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ \underline{1} & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_1 + 3\vec{r}_2 \\ \vec{r}_3 \mapsto \vec{r}_3 - 2\vec{r}_2}} \begin{bmatrix} 0 & 0 & 5 & 10 & -10 \\ 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & \underline{1} & 2 & -2 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_1 - 5\vec{r}_3 \\ \vec{r}_2 \mapsto \vec{r}_2 - 2\vec{r}_3}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_2 \mapsto \vec{r}_1}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_3 \\ \vec{r}_3 \mapsto \vec{r}_2}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the solution set to the homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ is

$$\operatorname{Ker}(\mathbf{A}) = \operatorname{Span}\left(\begin{bmatrix} 2 \\ \underline{1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ \underline{1} \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ \underline{1} \end{bmatrix} \right).$$

In the three vectors spanning this kernel, among entries corresponding to the free variables (namely the second, fourth, and fifth entries) there is a unique nonzero one. Hence, these three vectors are linearly independent and form a basis for the kernel. □

Exercises

9.0.9 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The empty set $\varnothing$ is a linear subspace of $\mathbb{K}^m$.
ii. Every linear subspace contains the zero vector.
iii. A hyperplane defines a linear subspace if and only if it contains the origin.
iv. The columns of a matrix form a basis for its columns space if and only if the matrix is invertible.

9.1 Dimension

HOW DO WE MEASURE THE SIZE OF A LINEAR SUBSPACE? To define this numerical invariant, we need a comparison result.

9.1.0 Lemma (Comparison)**.** *In any linear subspace, the number of vectors in any linearly independent set is less than or equal to the number of vectors in any spanning set.*

Proof. Let V be a linear subspace and fix two nonnegative integers n and m. Suppose that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in V are linearly independent and the vectors $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ span V. It follows that, for all $1 \leqslant k \leqslant n$, there exists scalars $a_{1,k}, a_{2,k}, \dots, a_{m,k} \in \mathbb{K}$ such that

$$\vec{v}_k = a_{1,k}\,\vec{w}_1 + a_{2,k}\,\vec{w}_2 + \cdots + a_{m,k}\,\vec{w}_m = \sum_{j=1}^{m} a_{j,k}\,\vec{w}_j\,.$$

For any vector $\vec{c} := \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^\mathsf{T} \in \mathbb{K}^n$, we obtain

$$\sum_{k=1}^{n} c_k \, \vec{v}_k = \sum_{k=1}^{n} c_k \left(\sum_{j=1}^{m} a_{j,k} \, \vec{w}_j \right) = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{j,k} \, c_k \right) \vec{w}_j \, .$$

Let $\mathbf{A}$ be the $(m \times n)$-matrix whose (j,k)-entry equals $a_{j,k}$. A nonzero solution $\vec{c} \in \mathbb{K}^n$ to the homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ produces a nonzero linear relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. If $m < n$, then this homogeneous linear system would have infinitely many solutions [6.0.4] and the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ would be linearly dependent, contradicting our assumption. Thus, we have $m \geqslant n$. □

9.1.1 Theorem (Equicardinality of bases)**.** *Any two bases of a linear subspace have the same number of vectors.*

The ***cardinality*** of a finite set is a nonnegative integer: the number of elements in the set.

Proof. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are both bases for a linear subspace V. Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent and the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ span V, Lemma 9.1.0 shows that $n \leqslant m$. Conversely, the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V and the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are linearly independent, so Lemma 9.1.0 implies that $m \geqslant n$. It follows that $m = n$. □

9.1.2 Definition. The ***dimension*** of a linear subspace V, denoted by $\dim(V)$, is the number of vectors in a basis.

9.1.3 Remarks.

- For any nonnegative integer n, the standard basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ establishes that $\dim(\mathbb{K}^n) = n$.
- The empty set $\varnothing$ is a basis for $\operatorname{Span}(\vec{\mathbf{0}})$, so $\dim \operatorname{Span}(\vec{\mathbf{0}}) = 0$.
- A linear subspace has dimension 1 if it is spanned by a single nonzero vector, so every 1-dimensional linear subspace is just a line through the origin.

Associated to any matrix are three linear subspaces: its row space, its column space, and its kernel. We want to relate the dimensions of these linear subspaces to other numerical invariants of the matrix. Better yet, we describe a preferred basis for each subspace.

9.1.4 Proposition. *The dimension of the row space of a matrix is equal to the rank of the matrix. Moreover, the nonzero rows in the reduced row echelon form of* $\mathbf{A}$ *form a basis for the row space of* $\mathbf{A}$*.*

Proof. By combining Propositions 9.0.3 and 9.0.7, we see that the nonzero rows in the reduced row echelon form of $\mathbf{A}$ for a basis for the row space. Since the rank of matrix is equal to the number of nonzero rows in its reduced row echelon form, the dimension of the row space equals the rank. □

9.1.5 Theorem (Rank–Nullity). *Let m and n be nonnegative integers. For any $(m \times n)$-matrix $\mathbf{A}$, we have $\operatorname{rank}(\mathbf{A}) + \dim\big(\operatorname{Ker}(\mathbf{A})\big) = n$. Moreover, the fundamental circuits associated to its reduced row echelon form are a basis for the kernel, the columns in $\mathbf{A}$ that contain leading ones in its reduced row echelon form are a basis for the column space of $\mathbf{A}$, and the dimension of the column space of $\mathbf{A}$ equals $\operatorname{rank}(\mathbf{A})$.*

The ***nullity*** of a matrix is the dimension of its kernel.

Proof. Let $\mathbf{B}$ be the reduced row echelon form of $\mathbf{A}$. Proposition 5.1.4 shows that the fundamental circuits associated to $\mathbf{B}$ span $\operatorname{Ker}(\mathbf{A})$. Among the fundamental circuits, there is a unique vector with a nonzero entry corresponding to a column without a leading one. Hence, the fundamental circuits are linearly independent and form a basis for the kernel of the matrix $\mathbf{A}$.

The columns of the $(m \times n)$-matrix $\mathbf{B}$ have a natural bipartition: those columns that contain a leading one and those that do not. The first paragraph shows that the columns that do not contain a leading one enumerate a basis for $\operatorname{Ker}(\mathbf{A})$. The complementary set consists of the columns containing a leading one, so this set has cardinality $\operatorname{rank}(\mathbf{A})$. Since there are n columns, we deduce that $\operatorname{rank}(\mathbf{A}) + \dim\big(\operatorname{Ker}(\mathbf{A})\big) = n$.

The characterizations [6.2.4] of a unique solution to a homogeneous linear system establishes that the columns in $\mathbf{A}$ that contain leading ones in $\mathbf{B}$ are linearly independent. Since the fundamental circuit associated to a column without a leading one, expresses this column as a linear combination of the columns in $\mathbf{A}$ that do contain leading entries in $\mathbf{B}$, we see that these columns in $\mathbf{A}$ span the column space. Therefore, the columns in $\mathbf{A}$ that contain leading one in $\mathbf{B}$ form a basis for the column space of $\mathbf{A}$. $\square$

9.1.6 Problem. Given that the reduced row echelon form of the matrix $\mathbf{A}$ equals $\mathbf{B}$, where

$$\mathbf{A} := \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find bases for the canonical linear subspaces associated to $\mathbf{A}$ and $\mathbf{B}$.

Solution. Proposition 9.1.4 shows that the row space of both $\mathbf{A}$ and $\mathbf{B}$ have the vectors $[1\ \ 0\ \ 1\ \ 0\ \ 1]^\mathsf{T}$, $[0\ \ 1\ \ -5\ \ 0\ \ 4]^\mathsf{T}$, $[0\ \ 0\ \ 0\ \ 1\ \ -2]^\mathsf{T} \in \mathbb{K}^5$ as a basis. Theorem 9.1.5 establishes that the kernel of both $\mathbf{A}$ and $\mathbf{B}$ have the vectors $[-1\ \ 5\ \ 1\ \ 0\ \ 0]^\mathsf{T}$, $[-1\ \ -4\ \ 0\ \ 2\ \ 1]^\mathsf{T} \in \mathbb{K}^5$ as a basis, the column space of $\mathbf{B}$ has the vectors $[1\ \ 0\ \ 0\ \ 0]^\mathsf{T}$, $[0\ \ 1\ \ 0\ \ 0]^\mathsf{T}$, $[0\ \ 0\ \ 1\ \ 0]^\mathsf{T} \in \mathbb{K}^4$ as a basis, and the column space of $\mathbf{A}$ has the vectors $[3\ \ 1\ \ -5\ \ -2]^\mathsf{T}$, $[1\ \ 0\ \ -2\ \ -1]^\mathsf{T}$, $[1\ \ 0\ \ -5\ \ 2]^\mathsf{T} \in \mathbb{K}^4$ as a basis. $\square$

Exercises

9.1.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Every linear subspace has a unique basis.
ii. No linear subspace has dimension zero.
iii. The column space and a the row space of a matrix must have the same dimension.
iv. The column space of a matrix is equal to the column space of its reduced row echelon form.

9.1.8 Problem. Find all of the maximal linearly-independent subsets among the columns of the matrix

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

9.1.9 Problem. Find bases for row space, column space, and kernel of

$$\mathbf{M} := \begin{bmatrix} 2 & 6 & -2 & -10 & 2 & -1 \\ 1 & 3 & 1 & -1 & -1 & 0 \\ -3 & -9 & 2 & 13 & -2 & 1 \\ -2 & -6 & -2 & 2 & 2 & -1 \end{bmatrix}.$$

9.2 Coordinates

HOW DO WE DESCRIBE VECTORS IN A LINEAR SUBSPACE? The choice of a basis for a linear subspace imposes a coordinate system: scalars that uniquely determine any vector in the linear subspace.

9.2.0 Lemma (Unique linear combinations)**.** *The columns of a matrix are basis for its column space if and only if every vector in the column space can be expressed uniquely as a linear combination of the columns.*

Proof. Let $\mathbf{B}$ be an $(m \times n)$-matrix, let $\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \dots, \vec{\boldsymbol{b}}_n \in \mathbb{K}^m$ denote the columns in $\mathbf{B}$, and let $V := \operatorname{Span}(\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \dots, \vec{\boldsymbol{b}}_n)$ be the column space.

$\Rightarrow$: Suppose that the columns of $\mathbf{B}$ form a basis for V. Since the columns span V, every vector $\vec{\boldsymbol{v}}$ in V can be written as a linear combination of the column vectors $\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \dots, \vec{\boldsymbol{b}}_n$. If there are scalars $c_1, c_2, \dots, c_n \in \mathbb{K}$ and $d_1, d_2, \dots, d_n \in \mathbb{K}$ such that

$$\vec{\boldsymbol{v}} = c_1 \vec{\boldsymbol{b}}_1 + c_2 \vec{\boldsymbol{b}}_2 + \cdots + c_n \vec{\boldsymbol{b}}_n = d_1 \vec{\boldsymbol{b}}_1 + d_1 \vec{\boldsymbol{b}}_1 + \cdots + d_n \vec{\boldsymbol{b}}_n$$

then we have

$$\vec{\boldsymbol{0}} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}} = (c_1 - d_1)\, \vec{\boldsymbol{b}}_1 + (c_2 - d_2)\, \vec{\boldsymbol{b}}_2 + \cdots + (c_n - d_n)\, \vec{\boldsymbol{b}}_n .$$

Since the vectors $\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \dots, \vec{\boldsymbol{b}}_n$ are linearly independent, it follows that $c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n = 0$ and the two linear combinations are equal.

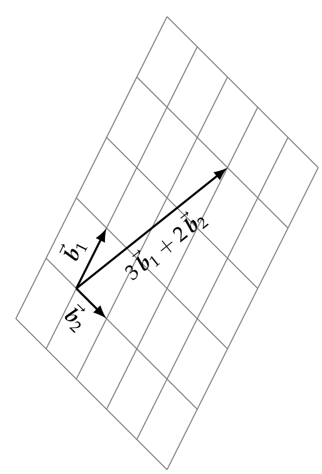

Figure 9.0: Visualizing nonstandard coordinates in the plane

$\Leftarrow$: The columns of $\mathbf{B}$ spans V, so every vector $\vec{v}$ in V is a linear combination of $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$. Since the zero vector $\vec{0}$ has a unique expression as a linear combination of the vectors $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$, these vectors are linearly independent. $\square$

9.2.1 Definition. Let $\mathbf{B}$ be an $(m \times n)$-matrix whose columns $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ form a basis for a linear subspace V. For any vector $\vec{v}$ in V, the ***coordinates of $\vec{v}$ relative to the basis*** $\mathbf{B}$ are the unique scalars $c_1, c_2, \dots, c_n \in \mathbb{K}$ such that $\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n$. We write $(\vec{v})_{\mathbf{B}} := [c_1 \;\; c_2 \;\; \cdots \;\; c_n]^{\mathsf{T}} \in \mathbb{K}^n$ for the ***coordinate vector*** of $\vec{v}$ relative to $\mathbf{B}$. In particular, we have $\mathbf{B}\,(\vec{v})_{\mathbf{B}} = \vec{v}$.

Since the columns of the identity matrix $\mathbf{I}$ form the standard basis of $\mathbb{K}^n$, it follows that, for any $\vec{v} \in \mathbb{K}^n$, we have

$$\vec{v} = v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + \cdots + v_n\,\vec{e}_n$$

and $(\vec{v})_{\mathbf{I}} = \vec{v}$.

9.2.2 Problem. Given that the columns of the matrix

$$\mathbf{B} := \begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix}$$

form a basis for its columns space, find the coordinates of the vector $\vec{v} := \begin{bmatrix} 3 & 12 & 7 \end{bmatrix}^{\mathsf{T}}$ relative to $\mathbf{B}$.

Solution. To solve the vector equation $c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{v}$, we find the reduced row echelon form of the augmented matrix:

$$\begin{bmatrix} 3 & -1 & 3 \\ \underline{6} & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_2 \mapsto (1/6)\,\vec{r}_2} \begin{bmatrix} 3 & -1 & 3 \\ \underline{1} & 0 & 2 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_1 - 3\,\vec{r}_2 \\ \vec{r}_3 \mapsto \vec{r}_3 - 2\,\vec{r}_2}} \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & 2 \\ 0 & \underline{1} & 3 \end{bmatrix}$$

$$\xrightarrow[\sim]{\vec{r}_1 \mapsto \vec{r}_1 + \vec{r}_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & \underline{1} & 3 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_2 \mapsto \vec{r}_1}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_3 \\ \vec{r}_3 \mapsto \vec{r}_2}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have $2\,\vec{b}_1 + 3\,\vec{b}_2 = \vec{v}$ and $(\vec{v})_{\mathbf{B}} = [2 \;\; 3]^{\mathsf{T}}$. Although the linear subspace $\operatorname{Span}(\vec{b}_1, \vec{b}_2)$ lies in $\mathbb{K}^3$, the vectors in this linear subspace are completely determined by just a pair of coordinates. $\square$

9.2.3 Proposition. *For any matrix* $\mathbf{B} := \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ *whose columns form a basis for a linear subspace V, the coordinate map from V to $\mathbb{K}^n$ defined by $\vec{x} \mapsto (\vec{x})_{\mathbf{B}}$ is a linear transformation: for all $\vec{v}, \vec{w} \in V$ and all $c, d \in \mathbb{K}$, we have $(c\,\vec{v} + d\,\vec{w})_{\mathbf{B}} = c\,(\vec{v})_{\mathbf{B}} + d\,(\vec{w})_{\mathbf{B}}$. Moreover, every vector in V corresponds to exactly one coordinate vector in $\mathbb{K}^n$.*

Proof. Suppose that

$$\begin{aligned} \vec{v} &= p_1\,\vec{b}_1 + p_2\,\vec{b}_2 + \cdots + p_n\,\vec{b}_n \quad &\Leftrightarrow \quad (\vec{v})_{\mathbf{B}} = \vec{p} = [p_1 \;\; p_2 \;\; \cdots \;\; p_n]^{\mathsf{T}} \in \mathbb{K}^n\,, \\ \vec{w} &= q_1\,\vec{b}_1 + q_2\,\vec{b}_2 + \cdots + q_n\,\vec{b}_n \quad &\Leftrightarrow \quad (\vec{w})_{\mathbf{B}} = \vec{q} = [q_1 \;\; q_2 \;\; \cdots \;\; q_n]^{\mathsf{T}} \in \mathbb{K}^n\,. \end{aligned}$$

Since the arithmetic of vectors is defined entrywise, we have

$$c\,\vec{v} + d\,\vec{w} = (c p_1 + d q_1)\,\vec{b}_1 + (c p_2 + d q_2)\,\vec{b}_2 + \cdots + (c p_n + d q_n)\,\vec{b}_n$$

which implies that

$$(c\,\vec{v} + d\,\vec{w})_{\mathbf{B}} = [cp_1 + dq_1 \;\; cp_2 + dq_2 \;\; \cdots \;\; cp_n + dq_n]^{\mathsf{T}} \in \mathbb{K}^n\,,$$
$$c\,(\vec{v})_{\mathbf{B}} + d\,(\vec{w})_{\mathbf{B}} = c\,\vec{p} + d\vec{q} = [cp_1 + dq_1 \;\; cp_2 + dq_2 \;\; \cdots \;\; cp_n + dq_n]^{\mathsf{T}} \in \mathbb{K}^n\,.$$

which proves linearity.

Every vector in $\mathbb{K}^n$ lies in the image, because $\vec{p} \in \mathbb{K}^n$ is the image of $\vec{v} = p_1\,\vec{b}_1 + p_2\,\vec{b}_2 + \cdots + p_n\,\vec{b}_n \in V$. The uniqueness of coordinates relative to a basis means that $\vec{p} = (\vec{v})_{\mathbf{B}} = (\vec{w})_{\mathbf{B}} = \vec{q}$ implies that $\vec{v} = \vec{w}$. Thus, we also deduce that every vector in V corresponds to a unique coordinate vector in $\mathbb{K}^n$. □

This coordinate map allows one to convert any problem about vectors in an arbitrary linear subspace into a problem about vectors in the coordinate space $\mathbb{K}^n$.

9.2.4 Corollary. *Let $\mathbf{B}$ be an $(m \times n)$-matrix whose columns $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ form a basis for a linear subspace V. For any positive integer k, the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in V are linearly independent if and only if the vectors $(\vec{v}_1)_{\mathbf{B}}, (\vec{v}_2)_{\mathbf{B}}, \ldots, (\vec{v}_k)_{\mathbf{B}}$ in $\mathbb{K}^n$ are linearly independent.*

Proof. For any scalars $c_1, c_2, \ldots, c_k \in \mathbb{K}$, the linearity of the coordinate map implies that $(\vec{0})_{\mathbf{B}} = \vec{0}$ and

$$(c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \cdots + c_k\,\vec{v}_k)_{\mathbf{B}} = c_1(\vec{v}_1)_{\mathbf{B}} + c_2(\vec{v}_2)_{\mathbf{B}} + \cdots + c_k(\vec{v}_k)_{\mathbf{B}}\,.$$

Hence, there exists a nonzero linear relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$ if and only if there is nonzero linear relation among the vectors $(\vec{v}_1)_{\mathbf{B}}, (\vec{v}_2)_{\mathbf{B}}, \ldots, (\vec{v}_k)_{\mathbf{B}} \in \mathbb{K}^n$. □

Exercises

9.2.5 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The number of coordinates for a linear subspace equals the dimension of the subspace.
ii. The number of coordinates for a linear subspace equals the dimension of its ambient space.
iii. Two distinct vectors may have the same coordinates relative to a given basis.
iv. Two distinct vectors may have the same coordinates relative to two different bases.

9.2.6 Problem. Consider the matrix $\mathbf{B} := \begin{bmatrix} 1 & -2 & 1 \\ -1 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix}$.

i. Show that the columns of the matrix $\mathbf{B}$ form a basis for $\mathbb{Q}^3$.
ii. Calculate the matrix $\mathbf{C} := \big[(\vec{e}_1)_{\mathbf{B}} \;\; (\vec{e}_2)_{\mathbf{B}} \;\; (\vec{e}_3)_{\mathbf{B}}\big]$, where $(\vec{e}_j)_{\mathbf{B}}$ is the coordinate vector of $\vec{e}_j \in \mathbb{Q}^3$ relative to the columns of $\mathbf{B}$.
iii. What is the relationship between $\mathbf{B}$ and $\mathbf{C}$?