# 10 Determinants



Surprisingly, the idea that a single number determines if a linear system has a unique solution predates the development of matrix theory. This chapter details the various properties that determine this number and focuses on a recursive method for calculating it.

## 10.0 Overview of Determinants

What is the most valuable scalar attached to a matrix? To every square matrix $\mathbf{A}$, we associate scalar called its determinant.

Although the term "determinant" was introduced into mathematics in 1802 by C.F. Gauss, A.-L. Cauchy first used the word "determinant" in 1812 as defined in this chapter.

**10.0.0 Strategy.** For each nonnegative integer $n$, the determinant $\det \colon \mathbb{K}^{n \times n} \to \mathbb{K}$ is a function from the set of all $(n \times n)$-matrices to the underlying field of scalars. There are many formulas for the determinant and all of them are complicated when $n$ is large. To understand the determinant, we employ the following steps.

- Choose one formula as the definition of the determinant.
- Show that the chosen formula satisfies certain key properties.
- Show that these key properties determine a unique function.
- Deduce other formulas by showing they satisfy the key properties.

Fix a nonnegative integer $n$ and consider an $(n \times n)$-matrix $\mathbf{A}$ whose $(j,k)$-entry is $a_{j,k} \in \mathbb{K}$ for all $\leqslant j \leqslant n$ and all $1 \leqslant k \leqslant n$. Let $\mathbf{A}(\hat{\jmath}, \hat{k})$ denote the $\big((n-1) \times (n-1)\big)$-submatrix of $\mathbf{A}$ obtained by deleting the $j$-th row and the $k$-th column.

**10.0.1 Definition** (Expansion along first row)**.** The ***determinant*** of the $(0 \times 0)$-matrix is 1. For all positive integers $n$, the ***determinant*** of the $(n \times n)$-matrix $\mathbf{A}$ is recursively defined by the formula

$$\begin{aligned} \det(\mathbf{A}) &:= a_{1,1} \det\big(\mathbf{A}(\hat{1},\hat{1})\big) - a_{1,2} \det\big(\mathbf{A}(\hat{1},\hat{2})\big) + \cdots + (-1)^{n+1} a_{1,n} \det\big(\mathbf{A}(\hat{1},\hat{n})\big) \\ &= \sum_{k=1}^{n} (-1)^{k+1} a_{1,k} \det\big(\mathbf{A}(\hat{1},\hat{k})\big) \,. \end{aligned}$$

**10.0.2 Remark.** This definition implies that the determinant of a $(1 \times 1)$-matrix is just the unique entry and recovers the formula for $(2 \times 2)$-matrices appearing in Problem 7.2.4; $\det\big([c]\big) = c(1) = c$ and

$$\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = a \det\big([d]\big) - c \det\big([b]\big) = ad - bc \,.$$

 

The next size of matrices also have an recognizable determinant; compare with Figure 2.9.

**10.0.3 Proposition.** *For any* $(3 \times 3)$*-matrix, we have*

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right) = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,1}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}\,.$$

*Proof.* The definition of the determinant gives

$$\begin{aligned} &\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right) \\ &= a_{1,1}\det\left(\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}\right) - a_{1,2}\det\left(\begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{bmatrix}\right) + a_{1,3}\det\left(\begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}\right) \\ &= a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} \\ &\qquad + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}\,. \end{aligned}$$
□

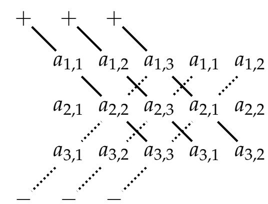

Figure 10.0: The determinant is the sum of the products along the solid diagonals minus the sum of the products along the dotted diagonals.

**10.0.4 Problem.** Compute $\det\left(\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -7 & 2 & 0 & 0 & 0 \\ 9 & -5 & 1 & 2 & 0 \\ 9 & 7 & 5 & 4 & -2 \\ 6 & 3 & 0 & -1 & 0 \end{bmatrix}\right)$.

*Solution.* We have

$$\begin{aligned} \det\left(\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -7 & 2 & 0 & 0 & 0 \\ 9 & -5 & 1 & 2 & 0 \\ 9 & 7 & 5 & 4 & -2 \\ 6 & 3 & 0 & -1 & 0 \end{bmatrix}\right) &= (3)\det\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 1 & 2 & 0 \\ 7 & 5 & 4 & -2 \\ 3 & 0 & -1 & 0 \end{bmatrix}\right) \\ &= (3)(2)\det\left(\begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & -2 \\ 0 & -1 & 0 \end{bmatrix}\right) \\ &= 6\left(\det\left(\begin{bmatrix} 4 & -2 \\ -1 & 0 \end{bmatrix}\right) - 2\det\left(\begin{bmatrix} 5 & -2 \\ 0 & 0 \end{bmatrix}\right)\right) \\ &= 6(-2) + 6(-2)(0) = -12\,. \end{aligned}$$
□

**10.0.5 Lemma** (Lower triangular determinants)**.** *For any lower triangular matrix, the determinant is the product of the diagonal entries.*

*Inductive proof.* Let $\mathbf{A}$ be an lower triangular $(n \times n)$-matrix. The base case $n = 0$ is vacuous, because the empty product is the multiplicative identity. Assume the claim holds for some nonnegative integer $n$. The definition of the determinant and the induction hypothesis give

$$\begin{aligned} \det(\mathbf{A}) = \det\left(\begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}\right) &= a_{1,1}\,\det\left(\begin{bmatrix} a_{2,2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{bmatrix}\right) \\ &= a_{1,1}(a_{2,2}\,a_{3,3}\,\cdots\,a_{n,n})\,. \end{aligned}$$
□

**10.0.6 Theorem** (Key determinantal properties)**.** *The determinant satisfies the following three properties.*

(identity) *The determinant of an identity matrix equals* 1.

(linearity) *The function* det *is linear in each column of the matrix.*

(vanishing) *When two adjacent columns are equal, the determinant is* 0.

*Proof.* Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{K}^n$ be the columns of the $(n \times n)$-matrix $\mathbf{A}$.

(identity): Since the identity matrix $\mathbf{I}$ is a lower triangular with 1s on the diagonal, Lemma 10.0.5 establishes that $\det(\mathbf{I}) = 1$.

(linearity): We proceed by induction on $n$. The base case $n = 0$ is vacuous. Suppose that the $j$-th column of $\mathbf{A}$ is a linear combination of two vectors. More explicitly, we write $\vec{a}_j = c\,\vec{v} + d\,\vec{w}$ for some $\vec{v}, \vec{w} \in \mathbb{K}^n$ and $c, d \in \mathbb{K}$ and set

$$\begin{aligned}\mathbf{B} &:= \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{j-1} & \vec{v} & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\\ \mathbf{C} &:= \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{j-1} & \vec{w} & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix} .\end{aligned}$$

The induction hypothesis guarantees that the determinant of each $\big((n-1) \times (n-1)\big)$-matrix $\mathbf{A}(\hat{1}, \hat{k})$ is linear in the $j$-th column for all $j < k$ and the determinant of each $\big((n-1) \times (n-1)\big)$-matrix $\mathbf{A}(\hat{1}, \hat{k})$ is linear in the $(j-1)$-st column for all $j > k$. Hence, we obtain

$$\begin{aligned}\det(\mathbf{A}) &= \sum_{k=1}^{n} (-1)^{k+1}\, a_{1,k}\, \det\big(\mathbf{A}(\hat{1},\hat{k})\big)\\ &= \sum_{k=1}^{j-1} (-1)^{k+1}\, a_{1,k}\, \det\big(\mathbf{A}(\hat{1},\hat{k})\big) + (-1)^{j+1}\, a_{1,j}\, \det\big(\mathbf{A}(\hat{1},\hat{j})\big) + \sum_{k=j+1}^{n} (-1)^{k+1}\, a_{1,k}\, \det\big(\mathbf{A}(\hat{1},\hat{k})\big)\\ &= \sum_{k \neq j} (-1)^{k+1}\, a_{1,k} \Big(c\, \det\big(\mathbf{B}(\hat{1},\hat{k})\big) + d\, \det\big(\mathbf{C}(\hat{1},\hat{k})\big)\Big) + (-1)^{j+1}\,(c\,v_j + d\,w_j)\det\big(\mathbf{A}(\hat{1},\hat{j})\big)\\ &= c\left(\sum_{k=1}^{n} (-1)^{k+1}\, b_{1,k}\, \det\big(\mathbf{B}(\hat{1},\hat{k})\big)\right) + d\left(\sum_{k=1}^{n} (-1)^{k+1}\, b_{1,k}\, \det\big(\mathbf{C}(\hat{1},\hat{k})\big)\right)\\ &= c\, \det(\mathbf{B}) + d\, \det(\mathbf{C}) ,\end{aligned}$$

which proves thet the determinant is linear in the $j$-th column.

(vanishing): We again proceed by induction on $n$. When $n = 2$, we have $\begin{bmatrix} a & a \\ b & b \end{bmatrix} = ab - ba = 0$ for any $a, b \in \mathbb{K}$. Suppose that the $i$-th and $(i+1)$-st columns of $\mathbf{A}$ are equal. The induction hypothesis implies that at most two summands in the expansion are nonzero:

$$\begin{aligned}\det(\mathbf{A}) &= \sum_{k=1}^{n} (-1)^{k+1}\, a_{1,k}\, \det\big(\mathbf{A}(\hat{1},\hat{k})\big)\\ &= (-1)^{i+1}\Big(a_{1,i}\, \det\big(\mathbf{A}(\hat{1},\hat{\imath})\big) - a_{1,i+1}\, \det\big(\mathbf{A}(\hat{1},\widehat{i+1})\big)\Big) .\end{aligned}$$

Since the $i$-th and $(i+1)$-st rows are equal, we see that $a_{1,i} = a_{1,i+1}$ and $\mathbf{A}(\hat{1},\hat{\imath}) = \mathbf{A}(\hat{1},\widehat{i+1})$. Hence, we conclude that $\det(\mathbf{A}) = 0$ because the signs in the expansion alternate. □

**10.0.7 Remark.** Since the determinant is linear in *each* column, we have

$$\det\left(\begin{bmatrix} 1 & 2+5 & (2)(2) \\ 3 & 4+6 & (2)(1) \\ 2 & -1+3 & (2)(0) \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 & (2)(2) \\ 3 & 4 & (2)(1) \\ 2 & -1 & (2)(0) \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 5 & (2)(2) \\ 3 & 6 & (2)(1) \\ 2 & 3 & (2)(0) \end{bmatrix}\right)$$
$$= (2)\det\left(\begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 1 \\ 2 & -1 & 0 \end{bmatrix}\right) + (2)\det\left(\begin{bmatrix} 1 & 5 & 2 \\ 3 & 6 & 1 \\ 2 & 3 & 0 \end{bmatrix}\right).$$

*Exercises*

**10.0.8 Problem.** Determine which of the following statements are true. If a statement is false, then provide a counterexample.

*i.* The determinant is defined for every matrix.

*ii.* Multiplying a column of a matrix by a scalar has the effect of multiplying the determinant by the same scalar.

*iii.* The determinant of a square zero matrix is zero.

*iv.* For any nonnegative integer $n$ and any scalar $c \in \mathbb{K}$, we have $\det(c\,\mathbf{I}_n) = c^n$.

## 10.1 Properties of Determinants

WHAT PROPERTIES DOES THE DETERMINANT ENJOY? Relying only the key determinantal properties [10.0.6], we enumerate several useful consequences. Throughout, let $n$ be a nonnegative integer and let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{K}^n$ denote the columns of an $(n \times n)$-matrix $\mathbf{A}$.

**10.1.0 Proposition** (Zero column vanishing)**.** *When a matrix has a column of zeros, its determinant equals* $0$*.*

*Proof.* In the matrix $\mathbf{A}$, suppose that $\vec{a}_i = \vec{\mathbf{0}} = 0\,\vec{a}_i$ for some column index $1 \leqslant i \leqslant n$. Linearity of the determinant [10.0.6] in the $i$-th column gives

$$\begin{aligned} & \det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_{i-1}\ \ \vec{a}_i\ \vec{a}_{i+1}\ \vec{a}_{i+2}\ \cdots\ \vec{a}_n]\big) \\ =\ & \det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_{i-1}\ 0\,\vec{a}_i\ \vec{a}_{i+1}\ \vec{a}_{i+2}\ \cdots\ \vec{a}_n]\big) \\ =\ & (0)\det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_{i-1}\ \ \vec{a}_i\ \vec{a}_{i+1}\ \vec{a}_{i+2}\ \cdots\ \vec{a}_n]\big) = 0\,. \end{aligned}$$ □

**10.1.1 Lemma** (Invariance under adjacent column add)**.** *When a multiple of a column is added to an adjacent column, the determinant is unchanged.*

*Proof.* Linearity of the determinant in the $i$-th column and vanishing when adjacent columns are equal [10.0.6] yields

$$\begin{aligned} & \det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_i\ \ \vec{a}_{i+1} + c\,\vec{a}_i\ \ \vec{a}_{i+2}\ \vec{a}_{i+3}\ \cdots\ \vec{a}_n]\big) \\ =\ & \det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_i\ \ \vec{a}_{i+1}\ \ \vec{a}_{i+2}\ \vec{a}_{i+3}\ \cdots\ \vec{a}_n]\big) + c\det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_i\ \vec{a}_i\ \vec{a}_{i+2}\ \vec{a}_{i+3}\ \cdots\ \vec{a}_n]\big) \\ =\ & \det\big([\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_i\ \ \vec{a}_{i+1}\ \ \vec{a}_{i+2}\ \vec{a}_{i+3}\ \cdots\ \vec{a}_n]\big)\,. \end{aligned}$$

Thus, multiplying the $i$-th column by the scalar $c$ and adding it to the $(i+1)$-st column does not change the determinant. □

**10.1.2 Problem.** Compute $\det\left(\begin{bmatrix} -5 & 0 & -6 & 3 \\ -6 & 5 & 7 & -1 \\ 4 & -3 & -7 & 2 \\ 9 & -6 & 4 & -5 \end{bmatrix}\right)$.

*Solution.* After multiplying the fourth column by 2 and adding it to the third column, Lemma 10.1.1 gives

$$\det\left(\begin{bmatrix} -5 & 0 & -6 & 3 \\ -6 & 5 & 7 & -1 \\ 4 & -3 & -7 & 2 \\ 9 & -6 & 4 & -5 \end{bmatrix}\right) \xrightarrow[=]{\vec{c}_3 \mapsto \vec{c}_3 + 2\,\vec{c}_4} \det\left(\begin{bmatrix} -5 & 0 & 0 & 3 \\ -6 & 5 & 5 & -1 \\ 4 & -3 & -3 & 2 \\ 9 & -6 & -6 & -5 \end{bmatrix}\right) = 0\,,$$

because the determinant vanishing whenever two adjacent columns are equal [10.0.6]. ☐

**10.1.3 Lemma** (Negation under adjacent column swap)**.** *When two adjacent columns are interchanged, the determinant is multiplied by* $-1$*.*

*Proof.* Using Lemma 10.1.1 three times and linearity in the $(i+1)$-st column [10.0.6], we have

$$\begin{aligned}
&\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\vec{c}_{i+1} \mapsto \vec{c}_{i+1} - \vec{c}_i} &\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} - \vec{a}_i & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\vec{c}_i \mapsto \vec{c}_i + \vec{c}_{i+1}} &\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i + (\vec{a}_{i+1} - \vec{a}_i) & \vec{a}_{i+1} - \vec{a}_i & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
= &\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & \vec{a}_{i+1} - \vec{a}_i & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\vec{c}_{i+1} \mapsto \vec{c}_{i+1} - \vec{c}_i} &\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & -\vec{a}_i & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
= &(-1)\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & \vec{a}_i & \vec{a}_{i+2} & \vec{a}_{i+3} & \cdots & \vec{a}_n\end{bmatrix}\big)\,.
\end{aligned}$$

Thus, interchanging the $i$-th and $(i+1)$-st columns changes the sign of the determinant. ☐

**10.1.4 Proposition** (Negation under a column swap)**.** *When any two columns are interchanged, the determinant is multiplied by* $-1$*.*

*Proof.* Interchanging any two columns can be accomplished by interchanging an odd number of adjacent columns. To be more explicitly, we have

$$\begin{aligned}
&\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_{i+j} & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\substack{\vec{c}_i \mapsto \vec{c}_{i+1} \\ \vec{c}_{i+1} \mapsto \vec{c}_i}} &(-1)\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & \vec{a}_i & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_{i+j} & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
&\qquad \ddots \\
= &(-1)^j\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & \vec{a}_{i+2} & \cdots & \vec{a}_{i+j-1} & \vec{a}_{i+j} & \vec{a}_i & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\substack{\vec{c}_{i+j-1} \mapsto \vec{c}_{i+j-2} \\ \vec{c}_{i+j-2} \mapsto \vec{c}_{i+j-1}}} &(-1)^{j+1}\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+1} & \vec{a}_{i+2} & \cdots & \vec{a}_{i+j} & \vec{a}_{i+j-1} & \vec{a}_i & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
&\qquad \iddots \\
= &(-1)^{2j-1}\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+j} & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_i & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
= &(-1)\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+j} & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_i & \vec{a}_{j+1} & \cdots & \vec{a}_n\end{bmatrix}\big)\,.
\end{aligned}$$

 

Hence, interchanging the $i$-th and $(i+j)$-th columns changes the sign of the determinant. □

**10.1.5 Corollary** (Vanishing under column repetition)**.** *When two columns of a matrix are equal, its determinant is* 0*.*

*Proof.* Interchanging columns yields a matrix with two adjacent columns being equal, so the determinant vanishes [10.0.6]. □

**10.1.6 Proposition** (Invariance under column add)**.** *When a multiple of a column is added to another column, the determinant is unchanged.*

*Proof.* An elementary column addition operation can be performed by a column swap, an adjacent column addition, and then reversing the column swaps. More explicitly, to multiply $\vec{a}_i$ by the scalar $c$ and add it to $\vec{a}_{i+j}$, we do the following:

$$\begin{aligned}
&\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_{i+j} & \vec{a}_{i+j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\substack{\vec{c}_i \mapsto \vec{c}_{i+j-1} \\ \vec{c}_{i+j-1} \mapsto \vec{c}_i}}\ &(-1)\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+j-1} & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_i & \vec{a}_{i+j} & \vec{a}_{i+j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\vec{c}_{i+1} \mapsto \vec{c}_{i+1} + c\,\vec{c}_i}\ &(-1)\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_{i+j+1} & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_i & \vec{a}_{i+j} + c\,\vec{a}_i & \vec{a}_{i+j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
\xrightarrow[=]{\substack{\vec{c}_i \mapsto \vec{c}_{i+j-1} \\ \vec{c}_{i+j-1} \mapsto \vec{c}_i}}\ &(-1)^2\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_{i+j} + c\,\vec{a}_i & \vec{a}_{i+j+1} & \cdots & \vec{a}_n\end{bmatrix}\big) \\
=\ &\det\big(\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{i-1} & \vec{a}_i & \vec{a}_{i+1} & \cdots & \vec{a}_{i+j-2} & \vec{a}_{i+j-1} & \vec{a}_{i+j} + c\,\vec{a}_i & \vec{a}_{i+j+1} & \cdots & \vec{a}_n\end{bmatrix}\big)\,.
\end{aligned}$$

Therefore, adding $c$ times the $i$-th column to the $(i+j)$-th column does not change the determinant. □

Exploiting this property is frequently the most efficient way to evaluate a determinant.

**10.1.7 Problem.** Compute $\det\left(\begin{bmatrix}1 & -4 & 2 \\ -2 & 9 & -9 \\ -1 & 7 & 0\end{bmatrix}\right)$.

*Solution.* We have

$$\begin{aligned}
\det\left(\begin{bmatrix}1 & -4 & 2 \\ -2 & 9 & -9 \\ -1 & 7 & 0\end{bmatrix}\right) &\xrightarrow[=]{\substack{\vec{c}_2 \mapsto \vec{c}_2 + 4\,\vec{c}_1 \\ \vec{c}_3 \mapsto \vec{c}_3 - 3\,\vec{c}_1}} \det\left(\begin{bmatrix}1 & 0 & 0 \\ -2 & 1 & -5 \\ -1 & 3 & 2\end{bmatrix}\right) \\
&\xrightarrow[=]{\vec{c}_3 \mapsto \vec{c}_3 + 5\,\vec{c}_2} \det\left(\begin{bmatrix}1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 17\end{bmatrix}\right) = 17\,.
\end{aligned}$$
□

$$\det\left(\begin{bmatrix}1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & c & \cdots & 1 & \\ & & & & & \ddots \\ & & & & & & 1\end{bmatrix}\right) = 1$$

$$\det\left(\begin{bmatrix}1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1\end{bmatrix}\right) = -1$$

$$\det\left(\begin{bmatrix}1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & d & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1\end{bmatrix}\right) = d.$$

**10.1.8 Corollary.** *The determinants of the elementary matrices are*

$$\begin{aligned}
\det(\mathbf{I} + c\,\mathbf{E}_{j,k}) &= 1\,, \\
\det(\mathbf{I} + \mathbf{E}_{j,k} + \mathbf{E}_{k,j} - \mathbf{E}_{j,j} - \mathbf{E}_{k,k}) &= -1\,, \\
\det\big(\mathbf{I} + (d-1)\mathbf{E}_{j,j}\big) &= d\,.
\end{aligned}$$

*Moreover, the determinant of an elementary matrix* $\mathbf{R}$ *is invariant under taking the transpose:* $\det(\mathbf{R}) = \det(\mathbf{R}^\mathsf{T})$

*Proof.* Invariance under elementary column addition establishes the first, sign change under columns swaps establishes the second, and linearity of the $i$-th column establishes the third. □

**10.1.9 Corollary.** *For any elementary matrix* $\mathbf{R}$ *and any square matrix* $\mathbf{A}$, *we have* $\det(\mathbf{A}\,\mathbf{R}) = \det(\mathbf{A})\,\det(\mathbf{R})$.

*Proof.* Left multiplication by an elementary matrix is equivalent to the corresponding elementary column operation, so the proposition follows from the properties of determinants and Corollary 10.1.8. □

*Exercises*

**10.1.10 Problem.** Determine which of the following statements are true. If a statement is false, then provide a counterexample.

*i.* The determinant of a nonzero matrix is never zero.
*ii.* The determinant of a matrix is zero only if the matrix contains a column of zeros.
*iii.* Performing elementary column operations does not change the determinant.
*iv.* The determinant of a product of matrices equals the product of determinants if the matrix on the right is elementary.

**10.1.11 Problem.** Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 & 7 \\ 0 & 3 & 0 & 0 & -2 \\ -1 & -10 & 1 & 1 & 1 \\ 0 & 9 & 0 & 0 & 1 \end{bmatrix}.$$

## 10.2 Characterization of Determinants

How do we characterize the determinant? As outlined in our overall strategy [10.0.0], we can now use the key properties [10.0.6] to characterize the determinant function. First, we record a convenient factorization for a non-invertible matrix.

**10.2.0 Lemma.** *For any non-invertible matrix* $\mathbf{A}$, *there exists a matrix* $\mathbf{B}$ *whose last column is zero and elementary matrices* $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_\ell$ *such that* $\mathbf{A} = \mathbf{B}\,\mathbf{R}_1\,\mathbf{R}_2 \cdots \mathbf{R}_\ell$.

*Proof.* Let $\mathbf{B}^\mathsf{T}$ be the reduced row echelon form of the matrix $\mathbf{A}^\mathsf{T}$. By the row reduction algorithm [4.2.0], there exists elementary matrices $\widetilde{\mathbf{R}}_1, \widetilde{\mathbf{R}}_2, \dots, \widetilde{\mathbf{R}}_\ell$ such that $\mathbf{B}^\mathsf{T} = \widetilde{\mathbf{R}}_\ell\,\widetilde{\mathbf{R}}_{\ell-1} \cdots \widetilde{\mathbf{R}}_1\,\mathbf{A}^\mathsf{T}$. The properties of the transpose [5.2.7] establish that $\mathbf{A} = \mathbf{B}\,(\widetilde{\mathbf{R}}_\ell^\mathsf{T})^{-1}\,(\widetilde{\mathbf{R}}_{\ell-1}^\mathsf{T})^{-1} \cdots (\widetilde{\mathbf{R}}_1^\mathsf{T})^{-1}$. Since the transpose and inverse of an elementary matrix is again an elementary matrix [8.0.2], set $\mathbf{R}_j := (\widetilde{\mathbf{R}}_{\ell-j}^{\mathsf{T}})^{-1}$ for all $1 \leqslant j \leqslant \ell$ to obtain $\mathbf{A} = \mathbf{B}\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell$. Since the matrix $\mathbf{A}$ is not invertible, the characterizations of invertible matrices [8.0.3] imply that $\mathbf{A}^{\mathsf{T}}$ is not invertible. It follows that the bottom row of $\mathbf{B}^{\mathsf{T}}$ does not contain a leading one and the last column of $\mathbf{B}$ is zero. □

**10.2.1 Theorem** (Characterization of determinants)**.** *The determinant is the only function satisfying the key properties* [10.0.6]*. Moreover, a square matrix is invertible if and only if its determinant is nonzero.*

*Proof.* Let $\mathbf{A}$ be a square matrix. To prove the first part, we show that one can compute $\det(\mathbf{A})$ using only the key properties of the determinant and their consequences. We consider two cases.

- Suppose that $\mathbf{A}$ is invertible. It follows from our characterization of invertible matrices that the matrix $\mathbf{A}$ is a product of elementary matrices: $\mathbf{A} = \mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell$. Using the compatibility of determinants with multiplication by elementary matrices [10.1.9] yields

$$\begin{aligned}\det(\mathbf{A}) &= \det(\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell)\\ &= \det(\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_{\ell-1})\det(\mathbf{R}_\ell)\\ &\;\;\vdots\\ &= \det(\mathbf{R}_1)\det(\mathbf{R}_2)\cdots\det(\mathbf{R}_\ell)\,.\end{aligned}$$

  Since the key properties determine the determinants of elementary matrices [10.1.8] and they are nonzero, we are able to compute $\det(\mathbf{A})$ in this case and it is nonzero.
- Suppose that $\mathbf{A}$ is not invertible. Lemma 10.2.0 produces the factorization $\mathbf{A} = \mathbf{B}\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell$ and the compatibility of determinants with multiplication by elementary matrices [10.1.9] gives

$$\begin{aligned}\det(\mathbf{A}) &= \det(\mathbf{B}\,\mathbf{R}_1\,\cdots\,\mathbf{R}_\ell)\\ &= \det(\mathbf{B}\,\mathbf{R}_1\,\cdots\,\mathbf{R}_{\ell-1})\det(\mathbf{R}_\ell)\\ &\;\;\vdots\\ &= \det(\mathbf{B})\det(\mathbf{R}_1)\cdots\det(\mathbf{R}_\ell)\,.\end{aligned}$$

  Since the matrix $\mathbf{B}$ has a column of zeros, a consequence of the key properties [10.1.0] establishes that $\det(\mathbf{B}) = 0$. Therefore, we are able to compute $\det(\mathbf{A})$ in this case and it equals 0. □

Exploiting this technique of proof, we also obtain to two important properties of determinants.

**10.2.2 Theorem.** *Let $n$ be a nonnegative integer. For any $(n \times n)$-matrices $\mathbf{A}$ and $\mathbf{B}$, we have*

| | |
|---|---|
| (compatibility with transpose) | $\det(\mathbf{A}^{\mathsf{T}}) = \det(\mathbf{A})$ |
| (compatibility with multiplication) | $\det(\mathbf{B}\,\mathbf{A}) = \det(\mathbf{B})\det(\mathbf{A})$ |

*Proof.* We consider two cases.

- Suppose that $\mathbf{A}$ is invertible. The characterization of invertible matrices [8.0.3] shows that $\mathbf{A}$ is a product of elementary matrices: $\mathbf{A} = \mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell$. Combining the compatibility of determinants with multiplication by elementary matrices [10.1.9], the determinants of elementary matrices [10.1.8], and the commutativity of the multiplication of scalars gives

$$\begin{aligned}\det(\mathbf{A}) &= \det(\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell)\\ &= \det(\mathbf{R}_1)\det(\mathbf{R}_2)\cdots\det(\mathbf{R}_\ell)\\ &= \det(\mathbf{R}_1^\mathsf{T})\det(\mathbf{R}_2^\mathsf{T})\cdots\det(\mathbf{R}_\ell^\mathsf{T})\\ &= \det(\mathbf{R}_\ell^\mathsf{T})\det(\mathbf{R}_{\ell-1}^\mathsf{T})\cdots\det(\mathbf{R}_1^\mathsf{T})\\ &= \det(\mathbf{R}_\ell^\mathsf{T}\mathbf{R}_{\ell-1}^\mathsf{T}\cdots\mathbf{R}_1^\mathsf{T})\\ &= \det\big((\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell)^\mathsf{T}\big) = \det(\mathbf{A}^\mathsf{T})\,.\end{aligned}$$

  Similarly, the compatibility of determinants with multiplication by elementary matrices [10.1.9] gives

$$\begin{aligned}\det(\mathbf{B}\,\mathbf{A}) &= \det(\mathbf{B}\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell)\\ &= \det(\mathbf{B}\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_{\ell-1})\det(\mathbf{R}_\ell)\\ &\;\;\vdots\\ &= \det(\mathbf{B})\det(\mathbf{R}_1)\det(\mathbf{R}_2)\cdots\det(\mathbf{R}_\ell)\\ &= \det(\mathbf{B})\det(\mathbf{R}_1\,\mathbf{R}_2)\det(\mathbf{R}_3)\cdots\det(\mathbf{R}_\ell)\\ &\;\;\vdots\\ &= \det(\mathbf{B})\det(\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell)\\ &= \det(\mathbf{B})\det(\mathbf{A})\,.\end{aligned}$$

- Suppose that $\mathbf{A}$ is not invertible. Having a nonzero determinants characterizes an invertible matrix, so $\det(\mathbf{A}) = 0$ and $\det(\mathbf{B})\det(\mathbf{A}) = 0$. Since the characterizations of invertible matrices [8.0.3] imply that $\mathbf{A}^\mathsf{T}$ is not invertible, we also have $\det(\mathbf{A}^\mathsf{T}) = 0$. Furthermore, Lemma 10.2.0 gives a factorization $\mathbf{A} = \mathbf{B}'\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell$ where the last column of the matrix $\mathbf{B}'$ is zero and $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_\ell$ are elementary matrices. It follows that the last column of the matrix product $\mathbf{B}\,\mathbf{B}'$ is zero, so $\det(\mathbf{B}\,\mathbf{B}') = 0$. Hence, the compatibility of determinants with multiplication by elementary matrices [10.1.9] yields

$$\begin{aligned}0 = \det(\mathbf{B}\,\mathbf{B}') &= \det(\mathbf{B}\,\mathbf{B}')\det(\mathbf{R}_1)\det(\mathbf{R}_2)\cdots\det(\mathbf{R}_\ell)\\ &= \det(\mathbf{B}\,\mathbf{B}'\,\mathbf{R}_1\,\mathbf{R}_2\,\cdots\,\mathbf{R}_\ell) = \det(\mathbf{B}\,\mathbf{A})\,.\end{aligned}$$ □

**10.2.3 Corollary.** *The key properties of the determinant and all of their consequences hold when the word "column" is replaced by the word "row".*

*Proof.* The map $\mathbf{A} \mapsto \mathbf{A}^\mathsf{T}$ interchanges rows and columns, so the claim follows the invariance of determinants under transposition. □

 

**10.2.4 Problem.** Show that $\det\left(\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 1 & 7 & 7 \\ 0 & 0 & 2 & 3 \\ 4 & 2 & 1 & 5 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 2 & 1 & 5 & 1 \\ 1 & 3 & 7 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 4 & 1 & 4 \end{bmatrix}\right)$.

*Solution.* Interchanging the first two columns and subtracting the third from the the fourth converts the left matrix into the right. Hence, the determinant is multiplied by $-1$. □

**10.2.5 Problem.** For any positive integer $n$, compute

$$\mathbf{D}_n := \det\left(\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}\right).$$

*Solution.* Subtract the first row from every other row and add $\frac{2}{k+1}$ times the $k$-th column to the first for all $2 \leqslant k \leqslant n-1$ to obtain

$$\mathbf{D}_n = \det\left(\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{bmatrix}\right) = \det\left(\begin{bmatrix} 3 + \frac{2}{3} + \frac{2}{4} + \cdots + \frac{2}{n} & 1 & 1 & 1 & \cdots & 1 \\ 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{bmatrix}\right).$$

Thus, we deduce that $D_n = n!\big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\big)$. □

*Exercises*

**10.2.6 Problem.** Determine which of the following statements are true. If a statement is false, then provide a counterexample.

*i.* The determinant of any product matrices equals the product of the determinants.

*ii.* The determinant of a matrix and its transpose are equal.

**10.2.7 Problem.** Determine all values of $\theta$ for which the matrix

$$\begin{bmatrix} \sin(\theta) & 3 & -\cos(\theta) \\ 0 & 2 & 0 \\ \cos(\theta) & -3 & \sin(\theta) \end{bmatrix}$$

is invertible.

**10.2.8 Problem.**

Let $n$ be a positive integer. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ be $(n \times n)$-matrices with $\mathbf{A}$ invertible.

*i.* Show that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A}) \det(\mathbf{D})$.

**Hint:** $\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$

*ii.* Find matrices $\mathbf{X}$ and $\mathbf{Y}$ which produce the block $\mathbf{L}\,\mathbf{U}$ factorization

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{X} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix} .$$

*iii.* Show that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A}) \det(\mathbf{D} - \mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{C})$.

*iv.* If $\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A}$ then prove that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A}\,\mathbf{D} - \mathbf{B}\,\mathbf{C})$.

*v.* If $\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A}$, then give an example such that

$$\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) \neq \det(\mathbf{A}\,\mathbf{D} - \mathbf{B}\,\mathbf{C}) .$$

**10.2.9 Problem.**

*i.* Find the determinants of the following matrices:

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} .$$

*ii.* For any positive integer $n$, use the results in part *i* to guess the determinant of the $(n \times n)$-matrix below. Confirm your guess by using properties of determinants and induction.

$$\mathbf{K}_n := \begin{bmatrix} 3 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 2 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{bmatrix} .$$