7
Inner Products

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Compared to the coordinate space $\mathbb{R}^n$, an abstract vector space lacks geometry. For example, it does not have the concept of the length of a vector. To regain some geometry, this chapter introduces an additional structure that associates a scalar to each pair of vectors.

7.0 Inner Products

How do we concoct an analogue of the dot product? The desired structure builds on the properties of the dot product.

7.0.0 Definition. An ***inner product*** on $\mathbb{C}$-vector space V is a function from $V \times V$ to $\mathbb{C}$, sending the pair $(\boldsymbol{v}, \boldsymbol{w})$ of vectors in $V \times V$ to the scalar $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ in $\mathbb{C}$, that satisfies the following four properties:

The axiomatic definition of an inner product was first given by Giuseppe Peano in 1898.

$$\begin{aligned}
&\text{(linearity)} && \langle c\,\boldsymbol{u} + d\,\boldsymbol{v}, \boldsymbol{w} \rangle = c\,\langle \boldsymbol{u}, \boldsymbol{w} \rangle + d\,\langle \boldsymbol{v}, \boldsymbol{w} \rangle && \text{for all } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{C}. \\
&\text{(conjugate-symmetry)} && \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{w}, \boldsymbol{v} \rangle} && \text{for all } \boldsymbol{v}, \boldsymbol{w} \text{ in } V. \\
&\text{(nonnegativity)} && \langle \boldsymbol{v}, \boldsymbol{v} \rangle \geqslant 0 && \text{for all } \boldsymbol{v} \text{ in } V. \\
&\text{(positivity)} && \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \text{ if and only if } \boldsymbol{v} = \mathbf{0}.
\end{aligned}$$

An ***inner product space*** is a vector space together with a specified inner product.

7.0.1 Remarks.

- For any fixed vector $\boldsymbol{w}$ in V, linearity asserts that the map from V to $\mathbb{C}$ defined by $\boldsymbol{v} \mapsto \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ is linear.
- Conjugate-symmetry guarantees that, for all vectors $\boldsymbol{v}$ in V, we have $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$ or equivalently $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \in \mathbb{R}$, so the nonnegative property is well-defined.
- When working over $\mathbb{R}$-vector spaces, conjugate-symmetry is just symmetry: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$ for all $\boldsymbol{v}$ and $\boldsymbol{w}$ in V.

Conventions differ as to which argument should be linear. In quantum mechanics and mathematical physics, one traditionally defines linearity in the second argument.

7.0.2 Notation. For any nonnegative integers m and n, the conjugate-transpose of a complex $(m \times n)$-matrix $\mathbf{A}$ is the $(n \times m)$-matrix $\mathbf{A}^\star := \overline{\mathbf{A}^\mathsf{T}} = (\overline{\mathbf{A}})^\mathsf{T}$.

Since conjugation and taking the transpose are both involutions, we have $(\mathbf{A}^\star)^\star = \mathbf{A}$ for all $\mathbf{A}$.

7.0.3 Problem. Let n be a positive integer and let $\mathbf{A}$ be an invertible complex $(n \times n)$-matrix. Show that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle := \boldsymbol{w}^\star\, \mathbf{A}^\star\, \mathbf{A}\, \boldsymbol{v}$ defines an inner product on the coordinate space $\mathbb{C}^n$.

Solution. For all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{C}^n$ and all scalars c, d in $\mathbb{C}$, the linearity of matrix multiplication gives

$$\begin{aligned}\langle c\,\boldsymbol{u} + d\,\boldsymbol{v}, \boldsymbol{w}\rangle &= \boldsymbol{w}^\star\,\mathbf{A}^\star\,\mathbf{A}\,(c\,\boldsymbol{u} + d\,\boldsymbol{v}) \\ &= c\,(\boldsymbol{w}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{u}) + d\,(\boldsymbol{w}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{v}) = c\,\langle \boldsymbol{u}, \boldsymbol{w}\rangle + d\,\langle \boldsymbol{v}, \boldsymbol{w}\rangle\ ,\end{aligned}$$

establishing the linearity property. The properties of the transpose and conjugation also give

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \boldsymbol{w}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{v} &= (\boldsymbol{w}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{v})^\mathsf{T} = \boldsymbol{v}^\mathsf{T}\,\mathbf{A}^\mathsf{T}\,(\mathbf{A}^\star)^\mathsf{T}\,(\boldsymbol{w}^\star)^\mathsf{T} \\ &= \overline{\boldsymbol{v}^\star\,\mathbf{A}^\star\,(\mathbf{A}^\star)^\star\,(\boldsymbol{w}^\star)^\star} = \overline{\boldsymbol{v}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{w}} = \overline{\langle \boldsymbol{w}, \boldsymbol{v}\rangle}\,,\end{aligned}$$

establishing conjugate-symmetry. Setting $\boldsymbol{w} := \mathbf{A}\,\boldsymbol{v}$, we obtain

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{v}\rangle &= \boldsymbol{v}^\star\,\mathbf{A}^\star\,\mathbf{A}\,\boldsymbol{v} = (\mathbf{A}\,\boldsymbol{v})^\star\,(\mathbf{A}\,\boldsymbol{v}) = \boldsymbol{w}^\star\,\boldsymbol{w} \\ &= \overline{w_1}w_1 + \overline{w_2}w_2 + \cdots + \overline{w_n}w_n = |w_1|^2 + |w_2|^2 + \cdots + |w_n|^2\ .\end{aligned}$$

We deduce the nonnegativity property, because the absolute value of any nonzero complex number is a positive real number. Since $\mathbf{A}$ is invertible, we see that $\boldsymbol{w} = \mathbf{A}\,\boldsymbol{v} = \mathbf{0}$ if and only if $\boldsymbol{v} = \mathbf{0}$, which establishes the positivity property. □

When the invertible matrix is just the identity matrix, we obtain the canonical inner product on the coordinate space $\mathbb{C}^n$. Similar constructions give the canonical inner products on the vector space of matrices and the vector space of continuous functions.

7.0.4 Definition. For any nonnegative integer n, the ***standard inner product*** on $\mathbb{C}^n$ is defined, for all $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{C}^n$, by

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle := \boldsymbol{w}^\star\,\boldsymbol{v} = \sum_{k=1}^{n} \overline{w_k}\,v_k = \overline{w_1}\,v_1 + \overline{w_2}\,v_2 + \cdots + \overline{w_n}\,v_n\,.$$

7.0.5 Definition. For any two nonnegative integers m and n, the ***Frobenius inner product*** on the $\mathbb{C}$-vector space $\mathbb{C}^{m \times n}$ of all complex $(m \times n)$-matrices is defined, for all matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{C}^{m\times n}$, by

$$\langle \mathbf{A}, \mathbf{B}\rangle_\mathrm{F} := \operatorname{tr}(\mathbf{B}^\star\,\mathbf{A}) = \sum_{j=1}^{m}\sum_{k=1}^{n} \overline{b}_{j,k}\,a_{j,k}\,.$$

This inner product is named after Georg Frobenius, seemingly inspired by his 1909 work on nonnegative matrices.

If the matrices are converted into column vectors, then this inner product coincides with the standard inner product.

7.0.6 Definition. The ***canonical inner product*** on the $\mathbb{R}$-vector space of all real-valued continuous functions on the interval $[a, b] \subset \mathbb{R}$ is defined, for all functions f and g, by $\langle f, g\rangle := \int_a^b f(t)\,g(t)\,dt$.

7.0.7 Remarks. For any continuous functions f, g, h on the interval $[a, b]$ and any scalars c, e in $\mathbb{R}$, the properties of definite integrals give

$$\langle c\,f + e\,g, h\rangle = \int_a^b (c\,f + e\,g)(t)\,h(t)\,dt = c\int_a^b f(t)\,h(t)\,dt + e\int_a^b g(t)\,h(t)\;dt = c\,\langle f, h\rangle + e\,\langle g, h\rangle \ ,$$

$$\langle f, g\rangle = \int_a^b f(t)\,g(t)\;dt = \int_a^b g(t) f(t)\;dt = \langle g, f\rangle \ ,$$

$$\langle f, f\rangle = \int_a^b f(t)\,f(t)\;dt = \int_a^b |f(t)|^2\;dt \geqslant 0\,.$$

For any continuous nonnegative function h such that $\int_a^b h(t)\;dt = 0$, it follows that $h(t) = 0$ for all $t \in [a, b]$. Hence, the equation $\langle f, f\rangle = 0$ implies that $|f(t)|^2 = 0$ and $f(t) = 0$ for all $t \in [a, b]$.

7.0.8 Definition. Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in an inner product space are ***orthogonal*** if $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0$.

7.0.9 Lemma (Properties of inner products)**.** *Let V be a complex inner product space.*
i. For all vectors $\boldsymbol{v}$ in V, we have $\langle \mathbf{0}, \boldsymbol{v}\rangle = \mathbf{0} = \langle \boldsymbol{v}, \mathbf{0}\rangle$.
i. For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in V and all scalars c, d in $\mathbb{C}$, we have

$$\langle \boldsymbol{u}, c\,\boldsymbol{v} + d\,\boldsymbol{w}\rangle = \bar{c}\,\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \bar{d}\,\langle \boldsymbol{u}, \boldsymbol{w}\rangle \ .$$

The zero vector is orthogonal to every vector. Moreover, the positivity of inner products implies that the zero vector is the only vector orthogonal to itself.

Proof. The properties of linear maps [3.0.3] and the linearity of inner products imply that $\langle \mathbf{0}, \boldsymbol{v}\rangle = \mathbf{0}$ and conjugate-symmetry implies that $\langle \boldsymbol{v}, \mathbf{0}\rangle = \overline{\langle \mathbf{0}, \boldsymbol{v}\rangle} = \mathbf{0}$. For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in V and all scalars c, d in $\mathbb{C}$, the same two properties also give

$$\begin{aligned}\langle \boldsymbol{u}, c\,\boldsymbol{v} + d\,\boldsymbol{w}\rangle &= \overline{\langle c\,\boldsymbol{v} + d\,\boldsymbol{w}, \boldsymbol{u}\rangle} \\ &= \bar{c}\,\overline{\langle \boldsymbol{v}, \boldsymbol{u}\rangle} + \bar{d}\,\overline{\langle \boldsymbol{w}, \boldsymbol{u}\rangle} = \bar{c}\,\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \bar{d}\,\langle \boldsymbol{u}, \boldsymbol{w}\rangle \ .\end{aligned}$$

□

Exercises

7.0.10 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.
i. The inner product of any two vectors is a real number.
ii. Inner products are not defined when the base field is $\mathbb{R}$.
iii. The $\mathbb{C}$-vector space $\mathbb{C}^n$ has only one inner product.
iv. If $\mathbb{C}^1$ is equipped with the standard inner product, then every nonzero vector is orthogonal to a unique vector.

7.0.11 Problem. Let V be a complex inner product space. For all vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in V, prove the following identities:

(polar identity) $$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \tfrac{1}{4}\left(\|\boldsymbol{v} + \boldsymbol{w}\|^2 - \|\boldsymbol{v} - \boldsymbol{w}\|^2 + \mathrm{i}\;\|\boldsymbol{v} + \mathrm{i}\,\boldsymbol{w}\|^2 - \mathrm{i}\;\|\boldsymbol{v} - \mathrm{i}\,\boldsymbol{w}\|^2\right)$$

(parallelogram identity) $$\|\boldsymbol{v} + \boldsymbol{w}\|^2 + \|\boldsymbol{v} - \boldsymbol{w}\|^2 = 2\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2\right)$$

7.1 Norms

HOW DO WE MEASURE LENGTH IN AN INNER PRODUCT SPACE? An inner product induces a notion of length.

7.1.0 Definition. Let V be an inner product space. For any vector $\boldsymbol{v}$ in V, its ***norm*** (also known as its ***length***) is $\|\boldsymbol{v}\| := \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} \in \mathbb{R}$.

To define a normed linear space, one must assume that the underlying field of scalars is either $\mathbb{R}$ or $\mathbb{C}$.

7.1.1 Lemma (Properties of norms)**.** *Let V be an inner product space.*

i. *We have $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.*
ii. *For all vectors $\boldsymbol{v}$ in V and all scalars c, we have $\|c\,\boldsymbol{v}\| = |c|\,\|\boldsymbol{v}\|$.*

Proof.

i. The positivity of an inner product [7.0.0] asserts that $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$. Since the real number 0 has a unique square root, we have $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

ii. Linearity and conjugate-symmetry of an inner product [7.0.0] show that $\|c\,\boldsymbol{v}\|^2 = \langle c\,\boldsymbol{v}, c\,\boldsymbol{v} \rangle = (c\,\bar{c})\,\langle \boldsymbol{v}, \boldsymbol{v} \rangle = |c|^2\,\|\boldsymbol{v}\|^2$. Since the norm and the absolute value of any scalar are a nonnegative real numbers, taking square roots gives $\|c\,\boldsymbol{v}\| = |c|\,\|\boldsymbol{v}\|$. □

Since the square root symbol represents the unique nonnegative square root, we have $|c| = \sqrt{c^2}$ for all scalars c in $\mathbb{R}$.

7.1.2 Theorem (Pythagorean)**.** *For any two orthogonal vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in an inner product space, we have $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$.*

Proof. Since orthogonality [7.0.8] implies that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$, the linearity and conjugate-symmetry of an inner product [7.0.0] establish that

$$\begin{aligned} \|\boldsymbol{v} + \boldsymbol{w}\|^2 &= \langle \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \langle \boldsymbol{w}, \boldsymbol{v} \rangle + \langle \boldsymbol{w}, \boldsymbol{w} \rangle \\ &= \|\boldsymbol{v}\|^2 + \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \overline{\langle \boldsymbol{v}, \boldsymbol{w} \rangle} + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 . \end{aligned}$$ □

7.1.3 Definition. A set of vectors is ***orthonormal*** if the vectors are pairwise orthogonal and each vector has norm 1.

7.1.4 Problem. Let $V := C([-1,1])$ be the real inner product space consisting of continuous functions on the interval $[-1,1] \subset \mathbb{R}$ where $\langle f, g \rangle := \int_{-1}^{1} f(x)\,g(x)\;dx$. Demonstrate that the three polynomial functions $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{\sqrt{2}}\,x$, and $\frac{\sqrt{5}}{2\sqrt{2}}(3\,x^2 - 1)$ form an orthonormal set.

Solution. Since

$$\begin{aligned} \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle &= \int_{-1}^{1} \tfrac{1}{2}\;dx = \tfrac{1}{2}\big(1 - (-1)\big) = 1\,, \\ \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{\sqrt{3}}{\sqrt{2}}\,x \right\rangle &= \int_{-1}^{1} \tfrac{\sqrt{3}}{2}\,x\;dx = \tfrac{\sqrt{3}}{2}\left[\tfrac{1}{2}x^2\right]_{x=-1}^{x=1} = 0\,, \\ \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{\sqrt{5}}{2\sqrt{2}}(3\,x^2 - 1) \right\rangle &= \int_{-1}^{1} \tfrac{\sqrt{5}}{4}(3\,x^2 - 1)\;dx = \tfrac{\sqrt{5}}{4}\left[x^3 - x\right]_{x=-1}^{x=1} = \tfrac{\sqrt{5}}{4}\left[x(x-1)(x+1)\right]_{x=-1}^{x=1} = 0\,, \\ \left\langle \tfrac{\sqrt{3}}{\sqrt{2}}\,x, \tfrac{\sqrt{3}}{\sqrt{2}}\,x \right\rangle &= \int_{-1}^{1} \tfrac{3}{2}\,x^2\;dx = \tfrac{3}{2}\left[\tfrac{1}{3}\,x^3\right]_{x=-1}^{x=1} = \tfrac{1}{2}\big(1 - (-1)\big) = 1\,, \\ \left\langle \tfrac{\sqrt{3}}{\sqrt{2}}\,x, \tfrac{\sqrt{5}}{2\sqrt{2}}(3\,x^2 - 1) \right\rangle &= \int_{-1}^{1} \tfrac{\sqrt{15}}{4}(3\,x^3 - x)\;dx = \tfrac{\sqrt{15}}{4}\left[\tfrac{3}{4}\,x^4 - \tfrac{1}{2}\,x^2\right]_{x=-1}^{x=1} = 0\,, \\ \left\langle \tfrac{\sqrt{5}}{2\sqrt{2}}(3\,x^2 - 1), \tfrac{\sqrt{5}}{2\sqrt{2}}(3\,x^2 - 1) \right\rangle &= \int_{-1}^{1} \tfrac{5}{8}(3\,x^2 - 1)^2\;dx = \tfrac{5}{8}\left[\tfrac{9}{5}\,x^5 - 2\,x^3 + x\right]_{x=-1}^{x=1} = \tfrac{1}{4}(9 - 10 + 5) = 1\,, \end{aligned}$$

these functions form an orthonormal set. □

7.1.5 Corollary (Parseval identity)**.** *Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m$ be an orthonormal set of vectors in an inner product space. For all scalars $c_1, c_2, \dots, c_m$, we have $\|c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_m\, \boldsymbol{u}_m\|^2 = |c_1|^2 + |c_2|^2 + \cdots + |c_m|^2$.*

This result is named after Marc-Antoine Parseval who described a similar result in an infinite-dimensional vector space.

Induction proof. We proceed by induction on the cardinality m of the orthonormal set. The base case $m = 0$ is true because the empty sum of vectors is $\mathbf{0}$, $\|\mathbf{0}\| = 0$, and the empty sum of scalars is 0. Suppose that $m > 0$. Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m\}$ is an orthonormal set, the properties of an inner product [7.0.9] give

$$\begin{aligned}\langle c_1\, \boldsymbol{u}_1,\, c_2\, \boldsymbol{u}_2 + c_3\, \boldsymbol{u}_3 + \cdots + c_m\, \boldsymbol{u}_m \rangle &= c_1 \,\langle \boldsymbol{u}_1,\, c_2\, \boldsymbol{u}_2 + c_3\, \boldsymbol{u}_3 + \cdots + c_m\, \boldsymbol{u}_m \rangle \\ &= c_1\, \overline{c_2}\, \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle + c_1\, \overline{c_3}\, \langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle + \cdots + c_1\, \overline{c_m}\, \langle \boldsymbol{u}_1, \boldsymbol{u}_m \rangle \\ &= c_1\, \overline{c_2}\, 0 + c_1\, \overline{c_3}\, 0 + \cdots + c_1\, \overline{c_m}\, 0 = 0\,.\end{aligned}$$

Hence, the Pythagorean theorem [7.1.2], induction hypothesis, and properties of a norm [7.1.1] give

$$\begin{aligned}\|c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_m\, \boldsymbol{u}_m\|^2 &= \|c_1\, \boldsymbol{u}_1\|^2 + \|c_2\, \boldsymbol{u}_2 + c_3\, \boldsymbol{u}_3 + \cdots + c_m\, \boldsymbol{u}_m\|^2 \\ &= |c_1|^2 \|\boldsymbol{u}_1\|^2 + |c_2|^2 + |c_3|^2 + \cdots + |c_m|^2 \\ &= |c_1|^2 + |c_2|^2 + \cdots + |c_m|^2\end{aligned}$$

because $c_1 \boldsymbol{u}_1$ and $c_2\, \boldsymbol{u}_2 + c_3\, \boldsymbol{u}_3 + \cdots + c_m\, \boldsymbol{u}_m$ are orthogonal. □

7.1.6 Corollary. *An orthonormal set of vectors is linearly independent.*

Proof. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m$ be an orthonormal set of vectors in an inner product space. Suppose that there are scalars $c_1, c_2, \dots, c_m$ such that $c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_m\, \boldsymbol{u}_m = \mathbf{0}$. The Parseval identity [7.1.5] implies that $0 = \|c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_m\, \boldsymbol{u}_m\|^2 = |c_1|^2 + |c_2|^2 + \cdots + |c_m|^2$. For all $1 \leqslant k \leqslant m$, the real number $|c_k|^2$ is nonnegative, so we deduce that $|c_1| = |c_2| = \cdots = |c_m| = 0$ which implies that $c_1 = c_2 = \cdots = c_m = 0$. Therefore, the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m$ are linearly independent. □

7.1.7 Corollary (Orthonormal coordinates)**.** *Let $\mathcal{U} := (\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n)$ be an ordered orthonormal basis for an inner product space V. For any vector $\boldsymbol{v}$ in V, we have $\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle\, \boldsymbol{u}_1 + \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle\, \boldsymbol{u}_2 + \cdots + \langle \boldsymbol{v}, \boldsymbol{u}_n \rangle\, \boldsymbol{u}_n$. Equivalently, the coordinate vector of $\boldsymbol{v}$ relative to the orthonormal basis $\mathcal{U}$ is*

$$(\boldsymbol{v})_{\mathcal{U}} = \begin{bmatrix} \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle & \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle & \cdots & \langle \boldsymbol{v}, \boldsymbol{u}_n \rangle \end{bmatrix}^{\mathsf{T}}.$$

Proof. Since $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ is a basis for the vector space V, there exists scalars $c_1, c_2, \dots, c_n$ such that $\boldsymbol{v} = c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_n\, \boldsymbol{u}_n$. For each $1 \leqslant k \leqslant n$, the linearity of an inner product [7.0.0] gives

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle &= \langle c_1\, \boldsymbol{u}_1 + c_2\, \boldsymbol{u}_2 + \cdots + c_n\, \boldsymbol{u}_n, \boldsymbol{u}_k \rangle \\ &= c_1\, \langle \boldsymbol{u}_1, \boldsymbol{u}_k \rangle + c_2\, \langle \boldsymbol{u}_2, \boldsymbol{u}_k \rangle + \cdots + c_{k-1}\, \langle \boldsymbol{u}_{k-1}, \boldsymbol{u}_k \rangle + c_k\, \langle \boldsymbol{u}_k, \boldsymbol{u}_k \rangle + c_{k+1}\, \langle \boldsymbol{u}_{k+1}, \boldsymbol{u}_k \rangle + \cdots + c_n\, \langle \boldsymbol{u}_n, \boldsymbol{u}_k \rangle \\ &= c_1\, 0 + c_2\, 0 + \cdots + c_{k-1}\, 0 + c_k\, 1 + c_{k+1}\, 0 + \cdots + c_n\, 0 = c_k\,,\end{aligned}$$

because the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ are orthonormal. □