Problems 04

Due: Friday, 1 October 2021 before 17:00 EDT

1. Let n be a positive integer.

(i) For any two vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$, show that $\big| \|\vec{\mathbf{v}}\| - \|\vec{\mathbf{w}}\| \big| \leqslant \|\vec{\mathbf{v}} - \vec{\mathbf{w}}\|$.

(ii) Given vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$ such that, for all $\vec{\mathbf{u}} \in \mathbb{R}^n$, we have $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$, prove that $\vec{\mathbf{v}} = \vec{\mathbf{w}}$.

2. For any three vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$, the ***scalar triple product*** is defined to be $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \in \mathbb{R}$.

(i) Prove that $|\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$ is the volume of the parallelepiped formed by the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$.

(ii) Demonstrate that $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$.

(iii) Show that the geometric definition of the cross product satisfies the distributivity property.
Hint: Use Problem 4.1 (ii).

3. (i) Use vectors to show that a triangle that is inscribed in a circle and has a diameter as one of its sides must be a right-angled triangle.

(ii) Use the dot product to prove the law of cosines. Specifically, in any triangle with sides of length a, b, and c, demonstrate that $c^2 = a^2 + b^2 - 2\,a\,b\,\cos(\varphi)$, where φ is the angle between the sides of length a and b.

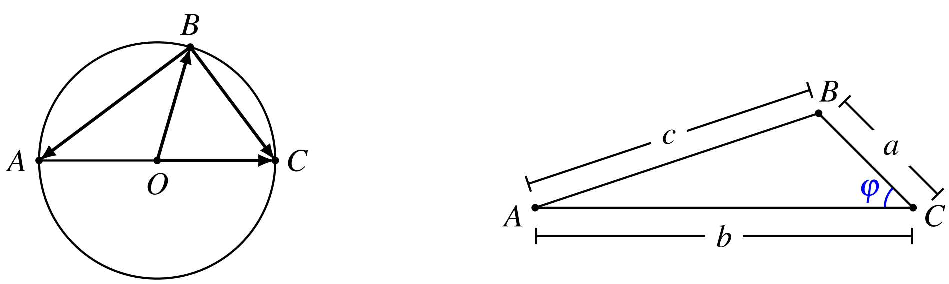

FIGURE 1. Two triangles