Problem Set #8  
MATH 110 : 2015–16  
Due: Friday, 6 November 2015

1. The **trace** of an \((n \times n)\)-matrix \(A := [a_{j,k}]\) is the sum of its diagonal entries:

\[
\text{tr}(A) := a_{1,1} + a_{2,2} + \cdots + a_{n,n} = \sum_{j=1}^{n} a_{j,j}.
\]

(a) Prove that the trace is linear. In other words, show that, for any \((n \times n)\)-matrices \(A, B\) and any scalar \(c \in \mathbb{K}\), we have \(\text{tr}(cA + B) = c \text{tr}(A) + \text{tr}(B)\).

(b) If \(A\) and \(B\) are \((n \times n)\)-matrices, then prove that \(\text{tr}(AB) = \text{tr}(BA)\).

(c) For \(n \geq 1\), show that \(XY - YX = I\) has no solutions for \((n \times n)\)-matrices \(X\) and \(Y\).

2. (a) Let \(P\) be an invertible \((m \times m)\)-matrix, let \(Q\) be an invertible \((n \times n)\)-matrix, let \(U\) be an \((m \times n)\)-matrix, and let \(V\) be an \((n \times m)\)-matrix. Directly verify the **Woodbury matrix identity**:

\[
(P + UQV)^{-1} = P^{-1} - P^{-1}U(Q^{-1} + VP^{-1}U)^{-1}VP^{-1}.
\]

(b) Let \(A\) be an invertible \((m \times m)\)-matrix, let \(B\) be an \((m \times n)\)-matrix, let \(C\) be an \((n \times m)\)-matrix, and let \(D\) be an \((n \times n)\)-matrix. If the **Schur complement** \(S := D - CA^{-1}B\) is invertible, then establish the blockwise inversion formula:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\
-S^{-1}CA^{-1} & S^{-1}
\end{bmatrix}.
\]

3. Consider the matrix \(M := \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ -1 & 0 & 1 \end{bmatrix}\).

(a) Verify that \(M^3 - 5M^2 + 9M - 4I = 0\).

(b) If \(N = \frac{1}{4}(M^2 - 5M + 9I)\), then prove \(N = M^{-1}\).

(c) Explain how \(N\) in part (b) can be obtained from the equation in part (a).