Problems 15

Due: Friday, 28 January 2022 before 17:00 EST

P15.1. Consider functions $g_1, g_2, \dots, g_n$ in the $\mathbb{R}$-vector space $C^{n-1}(\mathbb{R})$ of all real-valued functions on the real line whose $(n-1)$-st derivative exists and are continuous. The determinant of the $(n \times n)$-matrix

$$\begin{bmatrix} g_1(x) & g_2(x) & \cdots & g_n(x) \\ g_1'(x) & g_2'(x) & \cdots & g_n'(x) \\ g_1''(x) & g_2''(x) & \cdots & g_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(x) & g_2^{(n-1)}(x) & \cdots & g_n^{(n-1)}(x) \end{bmatrix}$$

is called the ***Wronskian***. When the Wronskian is nonzero at some point $x \in \mathbb{R}$, show that the functions $g_1, g_2, \dots, g_n$ are linearly independent.

P15.2. Let n be a nonnegative integer and let $a_0, a_1, \dots, a_n$ denote $n+1$ distinct real numbers. The ***Lagrange polynomials*** are defined, for all $0 \leqslant j \leqslant n$, by

$$\mathrm{L}_j(t) := \frac{(t-a_0)(t-a_1)\cdots(t-a_{j-1})(t-a_{j+1})(t-a_{j+2})\cdots(t-a_n)}{(a_j-a_0)(a_j-a_1)\cdots(a_j-a_{j-1})(a_j-a_{j+1})(a_j-a_{j+2})\cdots(a_j-a_n)} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{t-a_k}{a_j-a_k}.$$

(i) Compute the Lagrange polynomials when $n = 3$, $a_0 = 3$, $a_1 = 2$, $a_2 = 1$, and $a_3 = 0$.
(ii) Prove that the polynomials $\mathrm{L}_0, \mathrm{L}_1, \dots, \mathrm{L}_n$ form a basis for the $\mathbb{R}$-vector space $\mathbb{R}[t]_{\leqslant n}$.
(iii) Establish the Lagrange interpolation formula: for all $f(t) \in \mathbb{R}[t]_{\leqslant n}$, we have

$$f(t) = \sum_{j=0}^{n} f(a_j)\,\mathrm{L}_j(t) = f(a_0)\,\mathrm{L}_0(t) + f(a_1)\,\mathrm{L}_1(t) + \cdots + f(a_n)\,\mathrm{L}_n(t).$$

P15.3. For any square matrix $\mathbf{A}$ with entries in the field $\mathbb{K}$ of scalars, prove that there exists a nonzero polynomial p in $\mathbb{K}[t]$ such that $p(\mathbf{A}) = 0$.