1. The \( \mathbb{R} \)-vector space of trigonometric polynomials of degree at most \( n \) has the canonical ordered basis \( (1, \cos(x), \sin(x), \ldots, \cos(nx), \sin(nx)) \). For a fixed nonnegative real number \( a \in \mathbb{R} \), consider the linear operator \( T \) on the space of trigonometric polynomials of degree at most \( n \) defined by
\[
T[f] := \frac{d^2 f}{dx^2} + a^2 f
\]
for any trigonometric polynomial \( f \). For which \( a \) is the endomorphism \( T \) invertible?

2. Consider the following three complex \((2 \times 2)\)-matrices
\[
X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]
Problem \#15.3 shows that \( \mathcal{B} := (X, H, Y) \) is an ordered basis for \( \mathfrak{sl}(2) \), the subspace of traceless complex \((2 \times 2)\)-matrices. For a fixed matrix \( A \in \mathbb{C}^{2 \times 2} \), let \( \text{ad}_A : \mathfrak{sl}(2) \to \mathbb{C}^{2 \times 2} \) be the function defined by \( \text{ad}_A(B) = AB - BA \).

(a) Show that \( \text{ad}_A \) is a linear map.

(b) Show that the image of \( \text{ad}_A \) is contained in \( \mathfrak{sl}(2) \).

(c) Determine the matrices \( (\text{ad}_X)_B^B, (\text{ad}_H)_B^B \) and \( (\text{ad}_Y)_B^B \).

3. Let \( V = \mathbb{Q}^{n \times n} \) be the \( \mathbb{Q} \)-vector space of \((n \times n)\)-matrices and consider the linear operator \( T \in \text{End}(V) \) defined by \( T(A) := A^T \).

(a) Show that \( \pm 1 \) are the only eigenvalues of \( T \).

(b) Describe the eigenvectors corresponding to each eigenvalue of \( T \).

(c) Find an ordered basis \( \mathcal{C} \) for \( \mathbb{Q}^{2 \times 2} \) such that \( (T)_C^C \) is a diagonal matrix with respect to this basis.