Problems 21

Due: Friday, 18 March 2022 before 17:00 EST

P21.1. For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in a complex inner product space, prove the following:

$$\text{(polarization identity)} \qquad \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \tfrac{1}{4}\big(\|\boldsymbol{v}+\boldsymbol{w}\|^2 - \|\boldsymbol{v}-\boldsymbol{w}\|^2 + \mathrm{i}\,\|\boldsymbol{v}+\mathrm{i}\,\boldsymbol{w}\|^2 - \mathrm{i}\,\|\boldsymbol{v}-\mathrm{i}\,\boldsymbol{w}\|^2\big)\,,$$

$$\text{(parallelogram identity)} \quad 2\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2\right) = \|\boldsymbol{v}+\boldsymbol{w}\|^2 + \|\boldsymbol{v}-\boldsymbol{w}\|^2\,.$$

P21.2. Let n be a positive integer. Consider the ***maximum norm*** defined, for any vector $\boldsymbol{v}$ in $\mathbb{C}^n$, by

$$\|\boldsymbol{v}\|_\infty := \max\big(|v_1|, |v_2|, \dots, |v_n|\big)\,.$$

(i) Show that this norm satisfies the following four properties:
- (homogeneity) For any scalar c in $\mathbb{C}$ and any vector $\boldsymbol{v}$ in $\mathbb{C}^n$, we have $\|c\boldsymbol{v}\|_\infty = |c|\,\|\boldsymbol{v}\|_\infty$.
- (nonnegativity) For any vector $\boldsymbol{v}$ in $\mathbb{C}^n$, we have $\|\boldsymbol{v}\|_\infty \geqslant 0$.
- (positivity) We have $\|\boldsymbol{v}\|_\infty = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.
- (subadditivity) For all vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{C}^n$, we have $\|\boldsymbol{v}+\boldsymbol{w}\|_\infty \leqslant \|\boldsymbol{v}\|_\infty + \|\boldsymbol{w}\|_\infty$.

(ii) When $n \geqslant 2$, prove that this norm does not satisfy the parallelogram identity.

P21.3. For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in a complex inner product space, prove that

$$\|\boldsymbol{v}+\boldsymbol{w}\|\,\|\boldsymbol{v}-\boldsymbol{w}\| \leqslant \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2\,.$$

When does equality hold?