Problem Set #22  
MATH 110 : 2015–16  
Due: Friday, 18 March 2016

1. The population of Canada, as determined by the Canadian census, was as follows:

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>population (in millions)</td>
<td>27.3</td>
<td>28.8</td>
<td>30.0</td>
<td>31.6</td>
<td>33.5</td>
</tr>
</tbody>
</table>

Let \( t \) denote the time measured in years from 1991.
(a) Suppose that population (measure in millions) is modeled by the linear function \( p(t) = a + bt \). Find the least-squares estimates for the parameters \( a \) and \( b \).
(b) Suppose that the population (measure in millions) is modeled by the exponential function \( p(t) = ce^{\lambda t} \). Linearize the model and use the least-squares method to estimate the parameters \( c \) and \( \lambda \).

2. Fix \( n \in \mathbb{N} \) and let \( \mathcal{X} := \{ 2\pi\ell/n : 0 \leq \ell \leq n-1 \} \). Let \( V := \mathbb{C}^\mathcal{X} \) the complex inner product space, consisting of all functions from \( \mathcal{X} \) to \( \mathbb{C} \) with the inner product

\[
\langle f, g \rangle := \sum_{x \in \mathcal{X}} f(x)\overline{g(x)} = \sum_{\ell=0}^{n-1} f(2\pi\ell/n)\overline{g(2\pi\ell/n)}.
\]

(a) For all \( j \in \mathbb{Z} \) with \( 0 \leq j \leq n-1 \), show that the functions \( w_j(x) := e^{-j2\pi i x} \) are pairwise orthogonal and compute \( \|w_j\| \).
(b) For all \( k \in \mathbb{Z} \) with \( 0 \leq k \leq n-1 \), consider the function

\[
h_k(x) := \begin{cases} 
1 & \text{if } x = \frac{2\pi k}{n} \\
0 & \text{if } x \neq \frac{2\pi k}{n} 
\end{cases}.
\]

Which function in the linear subspace \( W := \text{Span}(w_0(x), w_1(x), \ldots, w_{n-1}(x)) \subset V \) best approximates the function \( h_k(x) \)?
(c) For all \( k \in \mathbb{Z} \) with \( 0 \leq k \leq n-1 \), calculate \( \|\text{proj}_W(h_k) - h_k\| \).

3. Let \( V \) be a finite-dimensional complex inner product space. Show that the adjoint operator on \( \text{End}(V) \) has the following four properties.
(conjugate-linear) For all \( S, T \in \text{End}(V) \) and for all \( c, d \in \mathbb{C} \), we have

\( (cS + dT)^* = \overline{c}S^* + \overline{d}T^* \).

(involutive) For all \( T \in \text{End}(V) \), we have \( (T^*)^* = T \).

(identity) For the identity operator \( I \in \text{End}(V) \), we have \( I^* = I \).

(multiplicative) For all \( S, T \in \text{End}(V) \), we have \( (ST)^* = T^*S^* \).