Problem Set #23
MATH 110 : 2015–16
Due: Thursday, 24 March 2016
Please submit your solutions in class on Thursday
or in the tutorial before 17:20!

1. (a) Prove that an orthogonal $(2 \times 2)$-matrix must have the form
\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\] or
\[
\begin{bmatrix}
a & b \\
b & -a
\end{bmatrix}
\]
where \( \begin{bmatrix} a \\ b \end{bmatrix} \) is a unit vector.

(b) Prove that an orthogonal $(2 \times 2)$-matrix must have the form
\[
\begin{bmatrix}
\cos(\theta) - \sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\] or
\[
\begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
\sin(\theta) - \cos(\theta)
\end{bmatrix}
\]
where $0 \leq \theta < 2\pi$.

2. Let $A$ and $B$ be $(n \times n)$-complex matrices. Suppose that $A$ and $B$ are simultaneously unitarily similar to upper-triangular matrices. In other words, there exists a unitary matrix $Q$ such that $Q^*AQ$ and $Q^*BQ$ are both upper-triangular matrices. Show that every eigenvalue of $AB - BA$ must be zero.

3. Consider the complex matrix
\[
A := \frac{1}{2} \begin{bmatrix}
1 + i & 1 - i \\
1 - i & 1 + i
\end{bmatrix}
\]
(a) Show that $A$ is normal.
(b) Find an orthonormal eigenbasis for $A$. 