1. Let $A$ be an $(n \times n)$-matrix with $n \geq 2$. If $\text{adj}(A)$ denotes the adjugate of $A$, then establish the following.

(a) We have $\det(\text{adj}(A)) = (\det(A))^{n-1}$.

(b) If $A$ is invertible, then we have $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2}A$.

(c) We have $\det(\text{adj}(\text{adj}(A))) = (\det(A))^{(n-1)^2}$.

**Solution.** The adjugate equation states that $\text{adj}(A)A = \det(A)I_n$.

(a) We consider three cases.

- If $A = 0$, then we have $\det(A) = 0$, $\text{adj}(A) = 0$, and $\det(\text{adj}(A)) = 0 = (\det(A))^{n-1}$.
- If $A \neq 0$ and $\det(A) = 0$, then the adjugate equation yields $\text{adj}(A)A = 0$, so the column space of $A$ is contained in the kernel of the matrix $\text{adj}(A)$. The characterization of invertible matrices shows that the matrix $\text{adj}(A)$ is not invertible. Hence, the characterization of determinants implies that $\det(\text{adj}(A)) = 0$ and we obtain $\det(\text{adj}(A)) = 0 = (\det(A))^{n-1}$.
- Suppose that $\det(A) \neq 0$. Since the determinant is multiplicative and linear in each row, the adjugate equation gives

$$\det(\text{adj}(A)) \det(A) = \det(\text{adj}(A)A) = \det(\det(A)I_n) = (\det(A))^{n} \det(I_n) = (\det(A))^{n}.$$ 

Dividing both sides by $\det(A)$ yields $\det(\text{adj}(A)) = (\det(A))^{n-1}$.

(b) The adjugate equation for the matrix $\text{adj}(A)$ is $\text{adj}(\text{adj}(A)) \text{adj}(A) = \det(\text{adj}(A))I_n$. Multiplying this equation on the right by the matrix $A$ and using part (a) gives

$$\det(A) \text{adj}(\text{adj}(A)) = \text{adj}(\text{adj}(A)) (\det(A)I_n) = \det(\text{adj}(A))A = (\det(A))^{n-1}A.$$ 

Since $A$ is invertible, the characterization of determinants establishes that $\det(A) \neq 0$. Hence, we may divide by $\det(A)$ to obtain $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2}A$.

(c) We again consider three cases.

- If $\text{adj}(A) = 0$, then the definition of the adjugate yields $\text{adj}(\text{adj}(A)) = 0$ and the adjugate equation gives $\det(A)I_n = \text{adj}(A)A = 0$, so we also obtain $\det(\text{adj}(A)) = 0$. It follows that

$$\det(\text{adj}(\text{adj}(A))) = 0 = (\det(A))^{(n-1)^2}.$$ 

- If $\text{adj}(A) \neq 0$ and $\det(A) = 0$, then part (a) together with the adjugate equation for $\text{adj}(A)$ imply that $\text{adj}(\text{adj}(A)) \text{adj}(A) = \det(\text{adj}(A))I_n = (\det(A))^{n-1}I_n = 0$, so the column space of $\text{adj}(A)$ is contained in the kernel of the matrix $\text{adj}(\text{adj}(A))$. The characterization of invertible matrices shows that the matrix $\text{adj}(\text{adj}(A))$ is not invertible. Hence, the characterization of determinants implies that $\det(\text{adj}(\text{adj}(A))) = 0$ and we obtain

$$\det(\text{adj}(\text{adj}(A))) = 0 = (\det(A))^{(n-1)^2}.$$ 

- Suppose that $\text{adj}(A) \neq 0$ and $\det(A) \neq 0$. Since the determinant is multiplicative and linear in each row, the adjugate equation for $\text{adj}(A)$ together with part (b) give

$$\det(\text{adj}(\text{adj}(A))) \det(\text{adj}(A)) = \det((\det(A))^{n-1}I_n) = (\det(A))^{(n-1)n}I_n.$$
It follows from part (a) that det(adj(A)) \neq 0, so dividing both sides by det(adj(A)) yields
\[
\det(\text{adj}(A)) = (\det(A))^{(n-1)n-(n-1)} = (\det(A))^{(n-1)^2}.
\]
\[\square\]

2. Let \( \delta_{j,k} \) denote \((j,k)\)-entry in the identity matrix \(I_n\). For \( n \in \mathbb{N} \), consider the \((n \times n)\)-matrix \( J \) in which the \((j,k)\)-entry equals \( 1 - \delta_{j,k} \). Compute \( \det(J) \).

\textbf{Solution.} Subtracting the first row from each of the other rows and adding \( k \)-th column to the first column for \( 2 \leq k \leq n \) yields

\[
\det(J) = \det \begin{bmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & 1 \\
1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{bmatrix} = \det \begin{bmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & -1 & 0 & \cdots & 0 \\
1 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & -1 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
n - 1 & 1 & \cdots & 1 & 1 \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Since the determinant of a triangular matrix is the product of its diagonal entries, we conclude that \( \det(J) = (-1)^{n-1}(n-1) \).
\[\square\]

3. Fix \( n \in \mathbb{N} \). For each \( 0 \leq k \leq 2n - 2 \), consider the polynomial \( p_k := c_1x_1^k + c_2x_2^k + \cdots + c_nx_n^k \) where \( c_1, c_2, \ldots, c_n \in \mathbb{K} \) are scalars and \( x_1, x_2, \ldots, x_n \) are variables. Show that

\[
\det \begin{bmatrix}
p_0 & p_1 & \cdots & p_{n-1} \\
p_1 & p_2 & \cdots & p_n \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_n & \cdots & p_{2n-2}
\end{bmatrix} = \left( \prod_{j=1}^{n} c_j \right) \left( \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \right).
\]

\textbf{Solution.} We first observe that

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{bmatrix} = \begin{bmatrix}
c_1 & 0 & \cdots & 0 \\
c_1x_1 & c_2 & \cdots & 0 \\
c_1x_1 & c_2x_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
c_1x_1 & c_2x_2 & \cdots & c_nx_n
\end{bmatrix} \begin{bmatrix}
1 & x_1^2 & \cdots & x_1^{n-1} \\
x_1 & x_2^2 & \cdots & x_2^{n-1} \\
x_1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & \cdots & x_n
\end{bmatrix} = \begin{bmatrix}
p_0 & p_1 & \cdots & p_{n-1} \\
p_1 & p_2 & \cdots & p_n \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_n & \cdots & p_{2n-2}
\end{bmatrix}.
\]
Since the determinant is multiplicative, the determinant of a triangular matrix is the product of its diagonal entries, and the determinant of the Vandermonde matrix is

\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
x_1 & x_2 & x_3 & \cdots & x_n \\
x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1}
\end{vmatrix}
= \begin{vmatrix}
1 & x_1^2 & \cdots & x_1^{n-1} \\
x_2 & x_2^2 & \cdots & x_2^{n-1} \\
x_3 & x_3^2 & \cdots & x_3^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_n & x_n^2 & \cdots & x_n^{n-1}
\end{vmatrix} = \prod_{1 \leq j < k \leq n} (x_k - x_j),
\]

we obtain the required formula. □