1. Let \( f_0(t) := 1, f_1(t) := t - 2, f_2(t) := t^2 + 3t - 9, \) and \( f_3(t) := t^3 - t^2 + 4t + 6. \)

(a) Show that \( \mathcal{B} := (f_0, f_1, f_2, f_3) \) is an ordered basis for \( \mathbb{R}[t]_{\leq 3}. \)

(b) Consider the equation \( a_0 + a_1 t + a_2 t^2 + a_3 t^3 = c_0 f_0(t) + c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) \)
where \( a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3 \in \mathbb{R}. \) If

\[
\vec{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4 \quad \text{and} \quad \vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{R}^4,
\]

then find matrices \( M \) and \( N \) such that \( M\vec{a} = \vec{c} \) and \( N\vec{c} = \vec{a}. \)

(c) Find the coordinates of \( t^2 \) and \( t^3 \) with relative to \( \mathcal{B}. \)

**Solution.**

(a) Since \( \dim \mathbb{R}[t]_{\leq 3} = 4, \) it is enough to establish that the polynomials \( f_0, f_1, f_2, f_3 \) are linearly independent. Let \( \mathcal{M} := (1, t, t^2, t^3) \) denote the monomial basis for \( \mathbb{R}[t]_{\leq 3}. \) The polynomials \( f_0, f_1, f_2, f_3 \) are linearly independent in \( \mathbb{R}[t]_{\leq 3} \) if and only if the vectors \( (f_0)_M, (f_1)_M, (f_2)_M, (f_3)_M \) are linearly independent in \( \mathbb{R}^4. \) Since

\[
(f_0)_M = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (f_1)_M = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (f_2)_M = \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad (f_3)_M = \begin{bmatrix} 6 \\ 4 \\ 0 \\ -1 \end{bmatrix}
\]

and

\[
\det [(f_0)_M (f_1)_M (f_2)_M (f_3)_M] = \det \begin{bmatrix} 1 & -2 & -9 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \neq 0,
\]

it follows that \( (f_0)_M, (f_1)_M, (f_2)_M, (f_3)_M \) is linearly independent.

(b) The columns of change of basis matrix from \( \mathcal{B} \) to \( \mathcal{M} \) are \( (f_j)_M. \) Thus, for the polynomial \( g(t) := a_0 + a_1 t + a_2 t^2 + a_3 t^3, \) we have

\[
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = (g)_M = [(f_0)_M (f_1)_M (f_2)_M (f_3)_M] (g)_\mathcal{B} = \begin{bmatrix} 1 & -2 & -9 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = N\vec{c}.
\]

The change of basis matrix from \( \mathcal{M} \) to \( \mathcal{B} \) is the inverse of \( N. \) Since the algorithm for finding the inverse gives

\[
\begin{bmatrix} 1 & -2 & -9 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{R}_4 + \text{R}_3 \rightarrow \text{R}_4} \begin{bmatrix} 1 & -2 & -9 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{R}_3 + \text{R}_2 \rightarrow \text{R}_2} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{R}_2 \rightarrow \text{R}_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
Let \( N = \begin{bmatrix} 1 & -2 & -9 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) so we obtain
\[
N = \begin{bmatrix} 1 & -2 & -9 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N^{-1} = M = \begin{bmatrix} 1 & 2 & 3 & -11 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

(c) We have
\[
(t^2)_B = M(t^2)_M = \begin{bmatrix} 1 & 2 & 3 & -11 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (t^3)_B = M(t^3)_M = \begin{bmatrix} 1 & 2 & 3 & -11 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\]

2. Let \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) be \( n + 1 \) distinct real numbers. For \( 0 \leq j \leq n \), the \( j \)-th Lagrange polynomials is defined to be
\[
\ell_j(t) := \frac{(t-a_0)(t-a_1) \cdots (t-a_{j-1})(t-a_{j+1})}{(a_j-a_0)(a_j-a_1) \cdots (a_j-a_{j-1})(a_j-a_{j+1})} \cdots (t-a_n) = \prod_{k=0}^{n} \left( \frac{t-a_k}{a_j-a_k} \right) \in \mathbb{R}[t].
\]

(a) Compute the Lagrange polynomials associated to \( a_0 = 1, a_1 = 2, a_2 = 3 \).
(b) Prove the polynomials \( \ell_0, \ell_1, \ldots, \ell_n \) are a basis for \( \mathbb{R}[t]_{\leq n} \).
(c) Deduce the Lagrange interpolation formula: for all \( q(t) \in \mathbb{R}[t]_{\leq n} \), we have
\[
q(t) = \sum_{j=0}^{n} q(a_j) \ell_j(t).
\]

**Solution.**

(a) The Lagrange polynomials associated to numbers \( \{1, 2, 3\} \) are
\[
\ell_0(t) = \frac{(t-2)(t-3)}{(1-2)(1-3)} = \frac{1}{2} t^2 - \frac{5}{2} t + 3,
\]
\[
\ell_1(t) = \frac{(t-1)(t-3)}{(2-1)(2-3)} = -t^2 + 4 t - 3,
\]
\[
\ell_2(t) = \frac{(t-1)(t-2)}{(3-1)(3-2)} = \frac{1}{2} t^2 - \frac{3}{2} t + 1.
\]

(b) Since \( \dim \mathbb{R}[t]_{\leq n} = n + 1 \), it suffices to show that the polynomials \( \ell_0, \ell_1, \ldots, \ell_n \) are linearly independent. Suppose \( c_0 \ell_0(t) + c_1 \ell_1(t) + \cdots + c_n \ell_n(t) = 0 \) for some \( c_0, c_1, \ldots, c_n \in \mathbb{R} \). The definition of the Lagrange polynomials implies that
\[
\ell_j(a_k) = \delta_{j,k} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.
\]
Hence, evaluating at \( t = a_k \) yields \( 0 = c_0 \ell_0(a_k) + c_1 \ell_1(a_k) + \cdots + c_n \ell_n(a_k) = c_k \) for \( 0 \leq k \leq n \). Therefore, we conclude that \( c_0 = c_1 = \cdots = c_n = 0 \) and the polynomials \( \ell_0, \ell_1, \ldots, \ell_n \) form a basis for \( \mathbb{R}[t]_{\leq n} \).

(c) Consider a polynomial \( q(t) \in \mathbb{R}[t]_{\leq n} \). Since \( \mathbb{R}[t]_{\leq n} = \text{Span}(\ell_0, \ell_1, \ldots, \ell_n) \), there exists \( c_0, c_1, \ldots, c_n \in \mathbb{R} \) such that \( q(t) = c_0 \ell_0(t) + c_1 \ell_1(t) + \cdots + c_n \ell_n(t) \). Evaluating at \( t = a_k \), we have \( q(a_k) = c_0 \ell_0(a_k) + c_1 \ell_1(a_k) + \cdots + c_n \ell_n(a_k) = c_k \). Hence, we obtain \( q(t) = q(a_0) \ell_0(t) + q(a_1) \ell_1(t) + \cdots + q(a_n) \ell_n(a_n) \). \( \square \)

Remark. The Lagrange interpolation formula implies that

\[
\int_\alpha^\beta q(t) \, dt = q(a_0) \int_\alpha^\beta \ell_0(t) \, dt + q(a_1) \int_\alpha^\beta \ell_1(t) \, dt + \cdots + q(a_n) \int_\alpha^\beta \ell_n(t) \, dt.
\]

Suppose \( a_k := \alpha + k \frac{\beta - \alpha}{n} \) for \( 0 \leq k \leq n \). For \( n = 1 \), the equation (\( \dagger \)) yields the trapezoidal rule for evaluating the definite integral for a polynomial. For \( n = 2 \), (\( \dagger \)) yields Simpson’s rule for evaluating the definite integral of a polynomial.

3. Let \( V \) be a finite-dimensional vector space and consider \( S, T \in \text{End}(V) \).
   (a) Show that \( ST \) is invertible if and only if both \( S \) and \( T \) are invertible.
   (b) Prove that \( ST = I \) if and only if \( TS = I \).
   (c) Give an example illustrating that both (a) and (b) are false over an infinite-dimensional vector space.

Solution.
(a) \( \implies \): Suppose \( ST \) is invertible. Since \( ST \) is bijective, we have \( \text{Ker}(ST) = \{0\} \) and \( \text{Im}(ST) = V \). If the vector \( w \in V \) lies in the image of \( ST \), then there exists a vector \( v \in V \) such that \( S(T[v]) = (ST)[v] = w \) and \( w \in \text{Im}(S) \). Similarly, if the vector \( v \in V \) belongs to the kernel of \( T \), then we have

\[
(ST)[v] = S(T[v]) = S(0) = 0
\]

and \( v \in \text{Ker}(ST) \). Hence, we obtain \( \text{Im}(ST) \subseteq \text{Im}(S) \) and \( \text{Ker}(T) \subseteq \text{Ker}(ST) \), which implies that \( \text{Im}(S) = V \) and \( \text{Ker}(T) = \{0\} \). For a linear operator on a finite-dimensional vector space, being invertible is equivalent to being injective or being surjective. Therefore, the endomorphisms \( S \) and \( T \) are invertible.

\( \iff \): If \( S \) and \( T \) are both invertible, then we have

\[
(T^{-1}S^{-1})(ST) = T^{-1}IT = T^{-1}T = I \quad \text{and} \quad (ST)(T^{-1}S^{-1}) = SIS^{-1} = SS^{-1} = I.
\]

Therefore, the inverse of \( ST \) is \( T^{-1}S^{-1} \).

(b) By symmetry, it suffices to prove that \( ST = I \) implies that \( TS = I \). If \( ST = I \), then we have \( \text{Ker}(T) \subseteq \text{Ker}(ST) = \{0\} \) and \( T \) is injective. Since \( V \) is finite-dimensional, \( T \) is invertible and \( TS = TS(1) = TS(T^{-1}T) = T(ST)T^{-1} = TIT^{-1} = TT^{-1} = I \).

(c) Suppose that \( V = \mathbb{R}[t] \). Consider \( D, J \in \text{End}(V) \) defined by

\[
(D[f])(t) = f'(t) \quad \text{and} \quad (J[f])(t) = \int_0^t f(y) \, dy \quad \text{for all} \; f \in V.
\]
The Fundamental Theorem of Calculus shows that
\[(DJ[f])(t) = \frac{d}{dt} \int_0^t f(y) \, dy = f(t) \quad \text{and} \quad (JD[f])(t) = \int_0^t f'(y) \, dy = f(t) - f(0)\,.

which implies \(DJ = I\) and \(JD \neq I\). In particular, over an infinite-dimensional vector space, both parts (a) and (b) are false. \(\square\)