1. The canonical ordered basis for $\mathbb{C}$-vector space of trigonometric polynomials of degree at most 1 is $\mathcal{B} = (1, \cos(z), \sin(z))$. Consider the endomorphism $J$ on this vector spaces defined by

$$\begin{align*}
(J[f])(z) &= \int_0^\pi f(z - t) \, dt.
\end{align*}$$

Show that $J$ is diagonalizable and find an eigenbasis.

**Solution.** Since we have

$$\begin{align*}
(J[1])(z) &= \int_0^\pi 1 \, dt = \pi,
(J[\cos])(z) &= \int_0^\pi \cos(z - t) \, dt = [\sin(z - t)]_0^\pi = -\sin(z - \pi) + \sin(z) = 2\sin(z),
(J[\sin])(z) &= \int_0^\pi \sin(z - t) \, dt = [\cos(z - t)]_0^\pi = \cos(z - \pi) - \cos(z) = -2\cos(z),
\end{align*}$$

it follows that

$$\begin{align*}
(J)_{\mathcal{B}} &\equiv \begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}.
\end{align*}$$

The characteristic polynomial is

$$\det((tI - J)_{\mathcal{B}}) = \det \begin{bmatrix} t - \pi & 0 & 0 \\ 0 & t - 2 & 0 \\ 0 & -2 & t \end{bmatrix} = (t - \pi)(t^2 + 4) = (t - \pi)(t - 2i)(t + 2i),$$

so the eigenvalues of $J$ are $-2i$, $2i$ and $\pi$. Since the dimension of the $\mathbb{C}$-vector space of trigonometric polynomials of degree at most 1 is 3 and $J$ has three distinct eigenvalues, we see that $J$ is diagonalizable. Moreover, we have

$$\begin{align*}
(-2iI - J)_{\mathcal{B}} &= \begin{bmatrix} -2i - \pi & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & -2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \text{Ker}(-2iI - J) = \text{Span}(\cos(x) + i\sin(x)),
(2iI - J)_{\mathcal{B}} &= \begin{bmatrix} 2i - \pi & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \text{Ker}(2iI - J) = \text{Span}(\cos(x) - i\sin(x)),
(\pi I - J)_{\mathcal{B}} &= \begin{bmatrix} \pi - \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -2\pi \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Ker}(\pi I - J) = \text{Span}(1).
\end{align*}$$

Therefore, $\mathcal{C} := (\cos(x) + i\sin(x), \cos(x) - i\sin(x), 1) = (e^{ix}, e^{-ix}, 1)$ is an eigenbasis for $J$ and

$$\begin{align*}
(J)_{\mathcal{C}} &\equiv \begin{bmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & \pi \end{bmatrix}.
\end{align*}$$

\[\square\]

2. Consider the linear operator $T \in \text{End}(\mathbb{R}[t]_{\leq 2})$ defined by

$$\begin{align*}
(T[f])(t) &= (1 - t^2) f''(t) - tf'(t) + 2f(t).
\end{align*}$$

Show that $T$ is diagonalizable and find an eigenbasis.
Solution. Fix the monomial basis \( M := (1, t, t^2) \) for \( \mathbb{R}[t]_{\leq 2} \). Since we have
\[
(T[1])(t) = (1 - t^2)(0) - t(0) + 2(1) = 2 \\
(T[t])(t) = (1 - t^2)(0) - t(1) + 2(t) = t \\
(T[t^2])(t) = (1 - t^2)(2) - t(2t) + 2(t^2) = 2 - 2t^2
\]
it follows that
\[
(T)_M^M = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}.
\]
Because this matrix is upper-triangular, we deduce that the eigenvalues of \( T \) are \(-2, 1, 2\). Since \( \dim(\mathbb{R}[x]_{\leq 2}) = 3 \) and \( T \) has three distinct eigenvalues, we see that \( T \) is diagonalizable. Moreover, we have
\[
(-2I - T)_M^M = \begin{bmatrix} -4 & 0 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Ker}(-2I - T) = \text{Span}(1 - 2t^2)
\]
\[
(I - T)_M^M = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Ker}(I - T) = \text{Span}(t)
\]
\[
(2I - T)_M^M = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Ker}(2I - T) = \text{Span}(1).
\]
Therefore, \( C := (1 - 2t^2, t, 1) \) is an eigenbasis for \( T \) and
\[
(T)_C^C = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \]

3. (a) A square matrix in **strictly diagonally dominant** if the absolute value of each diagonal entry is greater than the sum of the absolute values of the remaining entries in that row. Prove that a strictly diagonally dominant matrix must be invertible.

(b) Sketch the Gerschgorin disks in the complex plane that contain the eigenvalues of the following matrix:
\[
B := \begin{bmatrix} 10 & 0 & -1 & 1 \\ -1 & 12i & -1 & 2 \\ -1 & 3 & 20 & 2 \\ 1 & 2i & 3 & -45 \end{bmatrix}.
\]

Solution.
(a) Let \( a_{j,k} \in \mathbb{C} \) for \( 1 \leq j \leq n \) and \( 1 \leq k \leq n \) be the entries in a matrix \( A \in \mathbb{C}^{n \times n} \). If \( A \) is strictly diagonally dominant, then we have \( a_{k,k} > \sum_{k \neq j} |a_{k,j}| \) for all \( 1 \leq k \leq n \). If \( r_k := \sum_{j \neq k} |a_{k,j}| \), then Gerschgorin’s Theorem implies that each eigenvalue of \( M \) lies in \( \bigcup_{k=1}^n \{ z \in \mathbb{C} : |z - a_{k,k}| \leq r_k \} \). The inequality \( a_{k,k} > \sum_{k \neq j} |a_{k,j}| = r_k \) establishes that 0 does not belong to this union. Since 0 is not an eigenvalue of \( A \), we conclude that \( A \) is invertible.
(b) Gershgorin’s Theorem, applied to the rows and columns of $B$, implies that the eigenvalues of $B$ are contained in
\[
\{ z \in \mathbb{C} : |z - 10| \leq 2 \} \cup \{ z \in \mathbb{C} : |z - 12i| \leq 4 \} \cup \{ z \in \mathbb{C} : |z - 20| \leq 6 \} \cup \{ z \in \mathbb{C} : |z + 45| \leq 6 \},
\]
and
\[
\{ z \in \mathbb{C} : |z - 10| \leq 3 \} \cup \{ z \in \mathbb{C} : |z - 12i| \leq 5 \} \cup \{ z \in \mathbb{C} : |z - 20| \leq 5 \} \cup \{ z \in \mathbb{C} : |z + 45| \leq 5 \}.
\]
Hence, eigenvalues lie in the intersection
\[
\{ z \in \mathbb{C} : |z - 10| \leq 2 \} \cup \{ z \in \mathbb{C} : |z - 12i| \leq 4 \} \cup \{ z \in \mathbb{C} : |z - 20| \leq 5 \} \cup \{ z \in \mathbb{C} : |z + 45| \leq 5 \}.
\]
which is represented by the blue region in Figure 1. Since 0 is not contained in this region, the matrix is invertible. □

**Remark.** The eigenvalues of $B$ (appearing as black diamonds in the figure) are approximately:
\[
\begin{align*}
9.919313312 - 0.01637594225i, & \quad 0.1136038817 + 12.15746213i, \\
-45.12846127 - 0.0861104090i, & \quad 20.09554408 - 0.05497515064i.
\end{align*}
\]