Winter Review Problems

1. If we assume that wind resistance is proportional to the square of velocity, then downward velocity v of a falling body is given by

$$v = \sqrt{\frac{g}{k}}\left(\frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\right)$$

where g is the acceleration due to gravity and k is a constant. Find the height h of the body above the surface of the earth as a function of time. Assume the body starts at a height h_0.

2. Find the following integrals

(a) $\displaystyle\int \frac{d\theta}{\cos^3(\theta)}$ **(b)** $\displaystyle\int z\sqrt{1-z^2}\,dz$ **(c)** $\displaystyle\int \cos\big(\ln(t)\big)\,dt$

(d) $\displaystyle\int \frac{\ln\big(\ln(y)\big)}{y\ln(y)}\,dy$ **(e)** $\displaystyle\int \frac{dw}{1+e^w}$ **(f)** $\displaystyle\int \ln(a^2+x^2)\,dx$

(g) $\displaystyle\int \frac{\sin(\theta)}{\cos^2(\theta)+\cos(\theta)-2}\,d\theta$ **(h)** $\displaystyle\int \frac{y^4+3y^3+2y^3+1}{y^2+3y+2}\,dy$ **(i)** $\displaystyle\int \frac{e^t}{e^{2t}+3e^t+2}\,dt$

(j) $\displaystyle\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$ **(k)** $\displaystyle\int_0^{1/2} \frac{2-8y}{1+4y^2}\,dy$ **(l)** $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^2}\,ds$

3. Derive the following formulas:

(a) $$\int x^n\cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$$

(b) $$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

4. Calculate the following integrals, if they converge.

(a) $\displaystyle\int_1^{\infty} \frac{1}{5x+2}\,dx$ **(b)** $\displaystyle\int_0^{\infty} te^{-t}\,dt$ **(c)** $\displaystyle\int_{-\infty}^{\infty} \frac{dz}{z^2+25}$

(d) $\displaystyle\int_2^{\infty} \frac{dw}{w\ln w}$ **(e)** $\displaystyle\int_0^{2} \frac{1}{\sqrt{4-x^2}}\,dx$ **(f)** $\displaystyle\int_0^{\pi} \frac{1}{\sqrt{y}}e^{-\sqrt{y}}\,dy$

5. Find the volume of the following two solids. The base of a solid is the region between the curve $y = 2\sqrt{\sin(x)}$ and the interval $0 \leqslant x \leqslant \pi$ on the x-axis. The cross sections perpendicular to the x-axis are

(a) equilateral triangles with bases running from the x-axis to the curve.

(b) squares with bases running from the x-axis to the curve.

6. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $x = 12(y^2 - y^3)$ and $x = 0$ about the line $y = \frac{8}{5}$.

7. Find volume of the solid generated by revolving the region in the first quadrant bounded by the curve $y = x^2$, $y = 0$ and $x = 1$ about the line $x = -1$.

8. While taking a walk along the road where you live, you accidentally drop your glove. You don't know where you dropped it. Suppose the probability density $p(x)$ for having dropped the glove x kilometers from home (along the road) is $p(x) = 2e^{-2x}$ for $x \geqslant 0$.

(a) What is the probability that you dropped it within 1 kilometer of home?

(b) At what distance y from home is the probability that you dropped it with y kilometers equal to 0.95?

9. An object is moving counterclockwise at a constant speed around the circle $x^2 + y^2 = 1$ where x and y are measured in meters. It completes one revolution every minute.

(a) What is its speed?

(b) What is its velocity vector 30 seconds after it passes the point $(1, 0)$? Does your answer change if the object is moving clockwise? Explain.

10. Find the length of the following curves.

(a) $x = \cos(e^t)$, $y = \sin(e^t)$ for $0 \leqslant t \leqslant 1$.

(b) $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$.

(c) $y = \ln(1 - x^2)$ from $x = 0$ to $x = 1/2$.

11. Which of the following sequences converge and which diverge? Give reasons for your answers.

(a) $a_n = 1 - \dfrac{1}{n}$ for $n \geqslant 1$

(b) $b_k = k - \dfrac{1}{k}$ for $k \geqslant 1$

(c) $c_i = \dfrac{(2i+3)!}{(i+1)!}$ for $i \geqslant 0$

(d) $d_j = \left((-1)^j + 1\right)\left(\dfrac{j+1}{j}\right)$ for $j \geqslant 1$

12. Find the sum of the infinite series

$$1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \cdots .$$

13. Determine whether the following infinite series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty}(\sqrt{2})^{1-n}$ **(b)** $\displaystyle\sum_{i=0}^{\infty}\frac{1+2^i+3^i}{5^i}$ **(c)** $\displaystyle\sum_{j=0}^{\infty} je^{-j^2}$

(d) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\sqrt{\ln(k)}}$ **(e)** $\displaystyle\sum_{m=3}^{\infty}\frac{1}{m(\ln m)[\ln(\ln m)]^2}$ **(f)** $\displaystyle\sum_{\ell=1}^{\infty}\frac{1}{\ell^2+\ell+1}$

(g) $\displaystyle\sum_{q=1}^{\infty}\frac{q^2-q}{q^4+2}$ **(h)** $\displaystyle\sum_{p=1}^{\infty}\frac{p+2^p}{p+3^p}$ **(i)** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n}{3n+2}$

(j) $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k k}{\sqrt{2^k+1}}$ **(k)** $\displaystyle\sum_{j=1}^{\infty}\frac{\cos(j\pi)}{j^{3/2}}$ **(l)** $\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m^2\cos(m)}{3^{m^2}}$

(m) $\displaystyle\sum_{q=1}^{\infty}\frac{(q+2)!}{3^q(q!)^2}$ **(n)** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot4\cdot7\cdots(3n-2)}$

14. Find the radius and interval of convergence for each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{nx^n}{2^n}$ **(b)** $\displaystyle\sum_{m=0}^{\infty}\frac{(-4)^m x^m}{\sqrt{2m+1}}$ **(c)** $\displaystyle\sum_{k=1}^{\infty}\frac{(2x-1)^k}{k^4+16}$ **(d)** $\displaystyle\sum_{j=1}^{\infty}\frac{j!x^j}{j^j}$

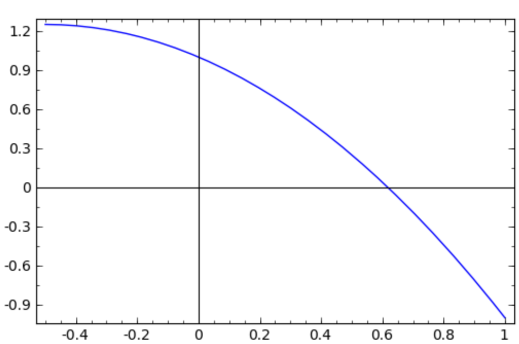
(e) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{n^2}$ **(f)** $\displaystyle\sum_{m=1}^{\infty}\frac{(\ln m)x^m}{3^m}$ **(g)** $\displaystyle\sum_{i=1}^{\infty}\frac{(-1)^{i+1}10^i(x-10)^{2i}}{i!}$ **(h)** $\displaystyle\sum_{k=0}^{\infty}\left(\frac{x^2+1}{5}\right)^k$

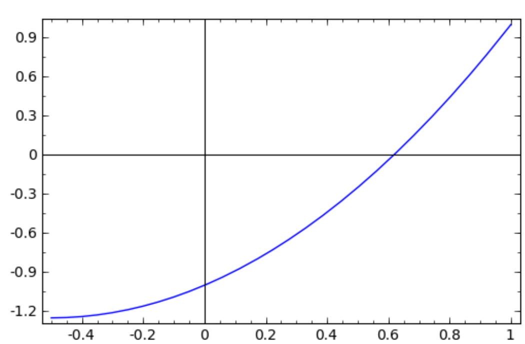
15. By recognizing each of the following series as a Taylor series evaluated at a particular value of x, find the sum of each of the following convergent series.

(a) $1+\frac{2}{1!}+\frac{4}{2!}+\frac{8}{3!}+\cdots+\frac{2^n}{n!}+\cdots$ **(b)** $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots+\frac{(-1)^n}{(2n+1)!}+\cdots$

(c) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots+\frac{1}{4^n}+\cdots$ **(d)** $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots$

16. Suppose $p(x)=a+bx+cx^2$ is the second degree Taylor polynomial for the function f about $x=0$. What can you say about the signs of a, b, c if f has the graph given below?





17. Find the first four nonzero terms of the Taylor series about 0 for each of the following functions.

(a) $\sqrt{1-2x}$ **(b)** $\ln(1-2y)$ **(c)** $\frac{z}{e^{z^2}}$ **(d)** $\sqrt{1+t}\sin(t)$

18. The Dubois formula relates a person's surface area s in m^2 to weight w in kg and height h in cm by $s = f(w,h) = \frac{1}{100}w^{1/4}h^{3/4}$. Find $f(65,160)$, $f_w(65,160)$ and $f_h(65,160)$. Interpret your answers in terms of surface area, height and weight.

19. Dieterici's equation of state for a gas is

$$P(V-b)e^{a/RVT} = RT\,, \qquad (\dagger)$$

where a, b and R are constants. Regard volume V as a function of temperature T and pressure P and prove that

$$\frac{\partial V}{\partial T} = \frac{R + \frac{a}{TV}}{\frac{RT}{V-b} - \frac{a}{V^2}}\,.$$

20. Let f be a differentiable function with $f_x(2,1) = -3$, $f_y(2,1) = 4$, and $f(2,1) = 7$.
(a) Give an equation for the tangent plane to the graph of f at $x = 2$, $y = 1$.
(b) Give an equation for the tangent line to the contour for f at $x = 2$, $y = 1$.

21. A function $u = f(x,y)$ with continuous second-order partial derivatives satisfying Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called a harmonic function. Which of the following functions are harmonic?

(a) $u(x,y) = x^3 - 3xy^2$ **(b)** $v(x,y) = x^2 + y^2$ **(c)** $w(x,y) = e^x\sin(y)$

22. The mean curvature of $f(x,y)$ is defined to be

$$\text{mean curvature} = \frac{(1+f_y^2)f_{xx} - 2f_{xy}f_xf_y + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}}\,.$$

Show that the function $f(x,y) = \arctan\left(\frac{y}{x}\right)$ has mean curvature equal to 0.

When you have completed all the problems (and not before), you may view my solutions at
http://www.mast.queensu.ca/~ggsmith/Math120/reviewWinterSolutions.pdf