Problem Set #8

Due: 31 October 2008

1. Suppose L is the line segment from the origin to the point $(4,12)$ and $\vec{\boldsymbol{F}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field defined by $\vec{\boldsymbol{F}}(x,y) := xy\vec{\boldsymbol{\imath}} + x\vec{\boldsymbol{\jmath}}$.

(a) Is line integral $\int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ greater than, less than, or equal to zero? Give a geometric explanation.

(b) A parameterization of L is $\vec{\boldsymbol{\gamma}} \colon [0,4] \to \mathbb{R}^2$ where $\vec{\boldsymbol{\gamma}}(t) := t\vec{\boldsymbol{\imath}} + 3t\vec{\boldsymbol{\jmath}}$. Use this to compute $\int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$.

(c) Suppose a particle leaves the point $(0,0)$, moves along the line towards the point $(4,12)$, stops before reaching it and backs up, stops again and reverses direction, then completes its journey to the endpoint. All travel takes place along the line segment joining the point $(0,0)$ to the point $(4,12)$. If we call this path L', explain why $\int_{L'} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$.

(d) A parameterization for a path like L' is given by $\vec{\boldsymbol{\beta}} \colon [0,4] \to \mathbb{R}^2$ with

$$\vec{\boldsymbol{\beta}}(t) = \tfrac{1}{3}(t^3 - 6t^2 + 11t)\vec{\boldsymbol{\imath}} + (t^3 - 6t^2 + 11t)\vec{\boldsymbol{\jmath}}.$$

Check that this parameterization begins at the point $(0,0)$ and ends at the point $(4,12)$. Also check that all points of L' lie on the line segment connecting the point $(0,0)$ to the point $(4,12)$. What are the values of t at which the particle changes direction?

(e) Find $\int_{L'} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ using the parameterization in part (d).

2. *This problem shows how the Fundamental Theorem for Line Integrals can be derived from the Fundamental Theorem for Calculus.* Suppose that $\vec{\boldsymbol{\gamma}} \colon [a,b] \to \mathbb{R}^3$ with $\vec{\boldsymbol{\gamma}}(t) := x(t)\vec{\boldsymbol{\imath}} + y(t)\vec{\boldsymbol{\jmath}} + z(t)\vec{\boldsymbol{k}}$ s a smooth parameterization of C with endpoints $\vec{\boldsymbol{p}} := \big(x(a), y(a), z(a)\big)$ and $\vec{\boldsymbol{q}} := \big(x(b), y(b), z(b)\big)$. Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth function.

(a) If $h \colon \mathbb{R} \to \mathbb{R}$ is the composite function $h(t) := f\big(\vec{\boldsymbol{\gamma}}(t)\big)$, then compute $h'(t)$.

(b) Show that $\int_C \vec{\boldsymbol{\nabla}} f \cdot d\vec{\boldsymbol{r}} = f(\vec{\boldsymbol{q}}) - f(\vec{\boldsymbol{p}})$.

3. **(a)** Consider the vector field $\vec{\boldsymbol{F}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ where $\vec{\boldsymbol{F}}(x,y,z) := y\vec{\boldsymbol{\imath}} + (3y^3 - x)\vec{\boldsymbol{\jmath}} + z\vec{\boldsymbol{k}}$. Evaluate $\int_{C_n} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ for the curve C_n parametrized by $\vec{\boldsymbol{\gamma}} \colon [0,1] \to \mathbb{R}^3$ where $\vec{\boldsymbol{\gamma}}(t) := t\vec{\boldsymbol{\imath}} + t^n\vec{\boldsymbol{\jmath}}$. What happens as $n \to \infty$?

(b) If H is the helix parametrized by $\vec{\boldsymbol{\varepsilon}} \colon [0, 1.25\pi] \to \mathbb{R}^3$ where

$$\vec{\boldsymbol{\varepsilon}}(t) := \cos(t)\vec{\boldsymbol{\imath}} + \sin(t)\vec{\boldsymbol{\jmath}} + t\,\vec{\boldsymbol{k}}\,,$$

then evaluate $\int_H yz^2e^{xyz^2}\,dx + xz^2e^{xyz^2}\,dy + 2xyze^{xyz^2}\,dz$.