Problem Set #6
Due: 22 October 2010

1. **(a)** For a vector field $\vec{\boldsymbol{F}}\colon \mathbb{R}^3 \to \mathbb{R}^3$, show that $\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) = 0$; in other words, the curl of a vector field is incompressible.

(b) If $f, g\colon \mathbb{R}^3 \to \mathbb{R}$ be scalar fields. If $\vec{\boldsymbol{G}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field defined by $\vec{\boldsymbol{G}} = f\vec{\boldsymbol{\nabla}}g$, then show $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{G}}$ is everywhere perpendicular to $\vec{\boldsymbol{G}}$.

2. Find a vector field $\vec{\boldsymbol{H}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}} = 2\vec{\boldsymbol{\imath}} - 3\vec{\boldsymbol{\jmath}} + 4\vec{\boldsymbol{k}}$.

Hint. Try $\vec{\boldsymbol{H}} := \vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}$ where $\vec{\boldsymbol{v}} \in \mathbb{R}^3$ is a fixed vector and the vector field $\vec{\boldsymbol{r}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\vec{\boldsymbol{r}}(x, y, z) := x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + z\vec{\boldsymbol{k}}$.

3. **(a)** If $p(x, y)$ gives the pollution density, in micrograms per square meter, and x and y are measured in meters, give the units and practical interpretation of $\int_R p(x, y)\, dA$.

(b) Using Riemann sums with four equal subdivisions in each direction, find upper and lower bounds for the volume under the graph of $h(x, y) = 2 + xy$ over the rectangle $R = [0, 2] \times [0, 4]$.