Problem Set #9

Due: 12 November 2010

1. **(a)** Suppose that $\vec{\boldsymbol{\gamma}} \colon [a,b] \to \mathbb{R}^3$ with $\vec{\boldsymbol{\gamma}}(t) := x(t)\vec{\boldsymbol{\imath}} + y(t)\vec{\boldsymbol{\jmath}} + z(t)\vec{\boldsymbol{k}}$ is a smooth parameterization of the curve C with endpoints $\vec{\boldsymbol{p}} := \big(x(a), y(a), z(a)\big)$ and $\vec{\boldsymbol{q}} := \big(x(b), y(b), z(b)\big)$. Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. If $h \colon \mathbb{R} \to \mathbb{R}$ is the composite function $h(t) := f\big(\vec{\boldsymbol{\gamma}}(t)\big)$, then compute $h'(t)$ and show that

$$\int_C \vec{\boldsymbol{\nabla}} f \cdot d\vec{\boldsymbol{r}} = f(\vec{\boldsymbol{q}}) - f(\vec{\boldsymbol{p}}) \,.$$

(b) If H is the helix parametrized by $\vec{\boldsymbol{\varepsilon}} \colon [0, 1.25\pi] \to \mathbb{R}^3$ where

$$\vec{\boldsymbol{\varepsilon}}(t) := \cos(t)\vec{\boldsymbol{\imath}} + \sin(t)\vec{\boldsymbol{\jmath}} + t\,\vec{\boldsymbol{k}}\,,$$

then evaluate $\displaystyle\int_H yz^2 e^{xyz^2}\,dx + xz^2 e^{xyz^2}\,dy + 2xyze^{xyz^2}\,dz$.

2. **(a)** If $\vec{\boldsymbol{F}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\vec{\boldsymbol{F}}(x,y) := x\vec{\boldsymbol{\jmath}}$, then show that the line integral of vector field $\vec{\boldsymbol{F}}$ around a closed curve in the xy-plane, oriented as in Green's Theorem, measures the area of the region enclosed by the curve.

(b) Calculate the area of the region within the folium of Descartes $x^3 + y^3 = 3xy$; it is parameterized by $\vec{\boldsymbol{\gamma}} \colon [0,\infty) \to \mathbb{R}^2$ where

$$\vec{\boldsymbol{\gamma}}(t) := \left(\frac{3t}{1+t^3}\right)\vec{\boldsymbol{\imath}} + \left(\frac{3t^2}{1+t^3}\right)\vec{\boldsymbol{\jmath}}.$$

3. Consider the vector field $\vec{\boldsymbol{F}} \colon \mathbb{R} \times (0,\infty) \to \mathbb{R}^2$ given by

$$\vec{\boldsymbol{F}}(x,y) := \frac{x + xy^2}{y^2}\vec{\boldsymbol{\imath}} - \frac{x^2+1}{y^3}\vec{\boldsymbol{\jmath}}.$$

(a) Determine if $\vec{\boldsymbol{F}}$ is path-independent.
(b) Find the work done by $\vec{\boldsymbol{F}}$ in moving a particle along the curve $y = 1 + x - x^2$ from $(0,1)$ to $(1,1)$.