Problem Set #9

Due: Friday, March 16, 2007

1. Suppose $S \in \mathrm{End}(V)$ satisfies $S^2 = S$. Prove that S is an orthogonal projection if and only if S is self-adjoint.

2. Suppose that $T \in \mathrm{End}(V)$ is self-adjoint, $\lambda \in \mathbb{K}$ and $\varepsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \varepsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \varepsilon$.

3. Define an inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]_{\leq 2}$ by $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$. Consider $J \in \mathrm{End}\big(\mathbb{R}[x]_{\leq 2}\big)$ defined by

$$(Jf)(x) = \frac{1}{8} \int_{-1}^{1} \big(7 + 6(x+t) - 6xt - 15(x^2 + t^2) + 45x^2t^2\big) f(t)\, dt\,.$$

(a) Show J is self-adjoint.
(b) Find an orthonormal eigenbasis for J.
(c) Find J^2, J^{-1} and $\cos(\pi J)$.

Hint. For part (c), use $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$.