Problem Set #10

Due: Friday, March 23, 2007

1. (a) Establish that any nonnegative linear combination of positive operators is positive.

(b) Suppose $T \in \operatorname{End}(V)$ is positive. Prove that T^k is positive for all positive integers k.

2. Let V be the $\mathbb{R}$-vector space of smooth functions f on the interval $[0, a]$ satisfying the boundary conditions $f(1) = f(a) = 0$. Define an inner product on V by $\langle f, g \rangle = \int_1^a f(x)g(x)e^x \, dx$. Consider the differential operator $D \in \operatorname{End}(V)$ given by $(Df)(x) = -e^{-x} \frac{d}{dx}\big(xf'(x)\big)$. Show that D is a positive operator.

3. (a) If $S \in \operatorname{End}(\mathbb{R}^3)$ is an isometry, then prove that there exists a nonzero vector $v \in \mathbb{R}^3$ such that $S^2 v = v$.

(b) Define $T \in \operatorname{End}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2)$. Find an isometry S such that $T = S\sqrt{T^*T}$.